LOW ORDER OBSERVER DESIGN VIA REALIZATION THEORY

By

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To my mother, Josefa, for being much more than what a mother should be; to Pura, mother-in-law and mother-in-deed, for her constant encouragement and support throughout my graduate studies; and to my wife, Lourdes, and my two sons, Jaime Jose and Gabriel, in lieu of the time I would have otherwise spent with them.

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LIST OF MATHEMATICAL SYMBOLS

Symbol	Usage	Meaning	First Usage
	A^T, $\underline{a}^T$	Transpose of A, $\underline{a}$	pg. 33
	A^{-1}	Inverse of A	pg. 16
#	$A^{\#}$	Pseudoinverse of A	pg. 47
ρ	$\rho(A)$	Rank of A	pg. 16
$\| \ \|$	$\|A\|$	Determinant of A	pg. 64
	$\mathrm{diag}(A_i)$	Block-diagonal matrix with block-elements A_i	pg. 145
$\neq$	$x \neq y$	x is not equal to y	pg. 28
$<$	$x < y$	x is less than y	pg. 27
$\leq$	$x \leq y$	x is less than or equal to y	pg. 27
$\subset$	$\mathcal{X} \subset \mathcal{Y}$	$\mathcal{X}$ is a subset of $\mathcal{Y}$	pg. 50
$\in$	$x \in \mathcal{X}$	x is an element of $\mathcal{X}$	pg. 30
$\rightarrow$	$\mathcal{X} \rightarrow \mathcal{Y}$	Map (set $\mathcal{X}$ to set $\mathcal{Y}$)	pg. 31
$\mapsto$	$x \mapsto x^2$	Map (element $x \in \mathcal{X}$ to element $x^2 \in \mathcal{Y}$)	pg. 32
$\cdot$	$A \cdot B$	Matrix multiplication operation	pg. 31
$\square$	$x \square y$	Abstract group operation	pg. 30
$\sim$	$x \sim y$	x is equivalent to y	pg. 30
$\{ \}$	$\{\cdot\}$	Sequence or set with elements $\cdot$	pg. 16
$\sum$		Summation	pg. 16
∞		Infinity	pg. 16
*		Unspecified element	pg. 21

LIST OF MATHEMATICAL SYMBOLS (Continued)

Symbol	Usage	Meaning	First Usage
!	n!	Factorial	pg. 16
$\forall$		For all	pg. 27
†		Footnote	pg. 16
∇		End of Proof, Remark, etc.	pg. 14

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The structure of linear constant finite-dimensional dynamic systems is studied. A complete system of invariants is obtained for the equivalence classes of completely controllable and observable systems defined by similarity transformations on the state-space. The realization of infinite and finite impulse response matrix sequences is considered in this framework. This leads to simple and efficient complete and partial minimal realization algorithms. One of the most significant properties of the algorithms given here is that the realizations are obtained in a canonical form. This allows a useful and straightforward parametrization of all possible minimal realizations in the finite impulse response matrix sequence case.

The situation often arises where it is convenient, possibly necessary, to realize a finite impulse response matrix sequence with a system which possesses certain properties (stability, arbitrary poles, cycli-

city, etc.), even though of non-minimal dimension. The questions of existence and parametrization of non-minimal realizations of finite impulse response matrix sequences are studied. Several significant new observations are made, and an algorithm is given to obtain all partial realizations of specified (non-minimal) dimension. The construction of stable partial realizations and partial realizations with arbitrary poles is also discussed. Specific attention is devoted to the question of minimality of these realizations, and several necessary and/or sufficient conditions for minimality are given.

The design of observers for linear constant finite-dimensional dynamic systems is considered. It is shown that an observer design problem can be converted in all cases to a partial realization problem. This result provides a new application for partial realization theory. The cases of observing a vector linear function of the state with a minimal order observer or a minimal order observer with arbitrary poles are specifically considered in the framework of the partial realization theory results developed here. Due to the nature of the partial realization algorithms, considerable information is gained about the structure and properties of all possible observers for a given vector linear function of the state. New, simple proofs are given for well-known results, and several new results are derived. It is shown how these results can be effectively used to design minimal order observers, minimal order observers with arbitrary poles, and intermediate order observers for a given vector linear function of the state. Some of the previous observer design techniques are examined in the light of partial realization theory,and their correspondence with other non-minimal partial realization algorithms is shown.

CHAPTER 1
INTRODUCTION

A considerable amount of the research effort in the last decade has been dedicated to studying the basic internal structure of linear dynamical systems, the so-called realization problem, and the design of observers. The research reported in this work is concerned mainly with realization theory and observer design, but considerable attention is also devoted to the structure of linear dynamical systems because any significant discussion of these two problems almost invariably involves the basic structure and properties of linear systems. At first glance, it may seem a bit unorthodox that realization theory and observer theory be discussed side-by-side, but it is shown that a realization problem evolves quite naturally from an observer design problem. This new result is not so surprising after one carefully examines the basic concepts of realization theory and observer theory.

A (complete) realization problem consists of processing the impulse response data of a linear dynamic system to obtain a model of the system in the form of several first-order linear differential (or difference) equations in the state of the system. Such models are compactly described by a triple of matrices: the system (or transition) matrix, the input matrix, and the output matrix. The usual formulation of the problem (and common sense) further requires that such a triple be of minimal size. The minimality requirement intuitively implies complete controllability and observability of the matrix triple; Kalman (1963) proved this intuitive notion to be true.

An observer is a linear dynamic system which processes the input/output information of a completely observable linear dynamic system and provides an asymptotic estimate of the state of the system which it observes. The estimate is termed asymptotic because the error between the state of the system and the output of the observer decreases exponentially to zero at a rate dictated by the dynamics of the observer. This estimate can be used, for example, to implement an (asymptotic) approximation of a state variable feedback control law for the observed system. Obviously, a desirable design objective is to construct the observer with the minimal number of dynamic components.

Consider now the above (rather loose) descriptions of a realization problem and an observer design problem. Notice that to solve a realization problem one has to develop a "machinery" (a series of mathematical operations) to process input/output data (impulse response data, to be more specific) of a system with the goal of obtaining an internal(state-space) description of the system. Notice also that, in a sense, an observer is a "machinery" which processes the input/output data of a known system and produces an estimate of the state (internal behavior) of the system which it observes. Further, minimality is a highly desirable quality for a solution to both a realization problem and an observer design problem. This heuristic discussion points out the similarity which exists between both concepts. Of course, a more rigorous development will be provided later on.

It is interesting to notice that the developments in realization theory and observer theory have occurred in parallel, with most of the results appearing in the last four or five years.

Survey of Previous Work in Realization Theory

Kalman (1963), Gilbert (1963), and Zadeh and Desoer (1963) were probably the first researchers to consider the realization problem in the modern control theory context. They formulated the problem as that of obtaining a minimal (completely controllable and observable) state-space representation of a system given the transfer function matrix. Gilbert (1963) and Zadeh and Desoer (1963) describe a realization procedure based on the rank of the residue matrices of the given transfer function. The dimension of the minimal realization is equal to the sum of the ranks of the residue matrices, and the realization is obtained in Jordan canonical form (see, for example, Gantmacher, 1959, Vol. 1). The multiplicity of a pole in the realization is given by the rank of the residue matrix at that pole. This procedure assumes that the poles in the given transfer function are of multiplicity one. The algorithm proposed by Kalman(1963) consists of realizing the given transfer function as a parallel combination of single-input, controllable subsystems (single-output, observable subsystems) in companion form, and then applying the so-called canonical structure theorem (Kalman, 1962) to delete the unobservable (uncontrollable) dynamics. This approach handles equally well simple and multiple poles in the transfer function.

At an earlier date, McMillan (1952) had discussed the network theory problem of obtaining the degree of a rational matrix. Kalman (1965) was the first to show that the realization problem as formulated in control theory is equivalent to the network theory formulation. The recently developed system matrix approach of Rosenbrock (1970) also gives considerable insight to the unification of the two lines of research.

In 1966, Ho and Kalman (1966) showed that the realization problem can be formulated and solved in the more tractable setting of matrix algebra. As before, the desired end result is a minimal state-space representation of the system, but now the infinite impulse response matrix sequence, rather than the transfer function matrix, is the starting information. From the impulse response data the so-called generalized Hankel array is formed, and the problem becomes one of studying the rank and null space of the Hankel array. Notice that given the impulse response matrix sequence instead of the transfer function matrix, the realization problem is more complicated because the minimal polynomial of the system is not available.

The work of Ho and Kalman (1966) gave new impetus to realization theory. Since then, several authors have dedicated their efforts to extend and/or improve the original results of Kalman (1963), Gilbert (1963), and Zadeh and Desoer (1963), and to provide alternate and/or improved realization algorithms based on the Hankel array formulation. Mayne (1968), Panda and Chen (1969), Roveda and Schmid (1970), Lal *et al.* (1972), and Huang (1974), among others, considered the older transfer function matrix approach, while Rissanen (1971, 1974), Ackermann and Bucy (1971), Silverman (1971), Chen and Mital (1972), Mital and Chen (1973), and Bonivento *et al.* (1973) approached the problem from the Hankel array formulation.

Rissanen (1974) and Dickinson *et al.* (1974a, b) have recently considered the problem of realizing a given infinite impulse response matrix sequence with a polynomial matrix pair. Such a polynomial matrix pair is referred to as a matrix-fraction description of the system, and

is becoming commonplace in the control literature, largely due to the ground work established by Popov (1969), Rosenbrock (1970), Wolovich (1972a, b, 1973a, b) and others.

The Hankel array formulation is used in this work. The realization algorithms of Rissanen (1971), Ackerman and Bucy (1971) and Mital and Chen (1972) are discussed in Chapter 2. A complete realization algorithm related to that of Bonivento et al. (1973) and Rissanen (1974) is also discussed at length in Chapter 2. Several significant points that they fail to indicate are brought to light in the discussion (the most significant being a recursiveness which is inherent in the algorithm).

Kalman (1971b)and Tether (1970) later considered the more realistic case where only a finite number of terms of the impulse response sequence are specified. This variant is commonly known as the partial realization problem, and corresponds (in the scalar case) to the Padé approximation problem of classical theory. The partial realization problem has several interesting features of its own, the most significant one being that a partial realization (whether minimal or not) is generally not unique.

The complete realization algorithms of Rissanen (1971) and Ackermann and Bucy (1971) (see Ackermann, 1972) can be easily adapted for use in the partial realization context. However, these two algorithms (as well as that of Kalman, 1971b,and Tether, 1970) are not very effective in recognizing and handling the degrees of freedom generally available in partial realizations.

One of the contributions of this dissertation is a partial realization algorithm which identifies the minimal number of realization parameters while at the same time conserves all the available degrees of

freedom. This property of the algorithm allows one to characterize all possible partial realizations (both minimal and non-minimal) in a simple form.

In the last few years, several intersting extensions have emerged from the original concept of realization theory. The work of Gopinath (1969), Budin (1971, 1972), Bonivento _et al._ (1973), and Audley and Rugh (1973) is concerned with obtaining a state-space representation given a general input/output map (not necessarily the impulse response) of a system. The results of Gopinath (1969), Budin (1971, 1972), and Bonivento _et al._ (1973) (see Bonivento and Guidorzi, 1971) can be generalized to the cases where the given data is contaminated by noise. In these cases, an estimate of the sytem parameters is obtained. Rissanen and Kailath (1972), Gupta and Fairman (1974), and Akaike (1974) have also considered the realization of stochastic systems. Both of these problems (realization from general input/output maps and realization of stochastic systems) show some of the overlap that exists between the research areas of realization theory and system identification theory.

Another interesting area which is currently receiving considerable attention is that of bilinear systems (nonlinear systems in which the nonlinearities enter the state differential equations as first order products of the states with the inputs). Isidori (1973) and Bruni _et al._ (1974), among others, have made extensive contributions in this area, particularly for the problem of constructing minimal bilinear realizations from nonlinear input/output maps.

Research in realization theory and its applications continues. The results discussed in this dissertation will hopefully guide the way for

new applications of realization theory, observer design theory, and other related fields. Chapter 5 presents some suggestions for future research.

Survey of Previous Work in Observer Design Theory

In 1964 Luenberger (1964) introduced observers as the deterministic counterparts of the statistical estimators of Kalman and Bucy (1961). (A statistical estimator is a system which processes the noise-corrupted input/output information of a system and provides a minimum variance estimate of the state of the system.) The structure of observers and statistical estimators is the same. However, they differ mainly in the following two considerations: a) the dynamics of an observer are practically arbitrary, while the dynamics of a statistical estimator are determined by the noise processes, and b) the dimension of an observer can be considerably less than the dimension of the system which it observes while the dimension of a statistical estimator is always equal to the dimension of the system which it estimates (provided all the available output measurements are corrupted by additive noise). These differences are sufficient to cause the respective design techniques to be considerably different. Only deterministic observers for linear, time-invariant systems are considered in this work.

Luenberger (1964) considered first the design of observers for single output systems only. He showed that an observer of dimension equal to the number of system states (n) minus the number of linearly independent system outputs (r) can be designed for multiple output systems, but did not propose a procedure to do so. He also showed that if the observer output is used in place of the unmeasurable state in a state

variable feedback control law, then the overall system-observer poles are the poles of the system with state variable feedback and the poles of the observer.

Two years later, Luenberger (1966) extended his original results and proposed a technique to design observers for multiple output systems. He further proved (by construction) that the dimension of the observer can be as low as $\nu - 1 \leq n - r$ where ν is the observability index of the observed system (see Definition (2.18)), if a scalar linear function of the state, rather than the complete state, is to be observed.

Since then, several researchers have undertaken the task of studying the properties of observers and pursuing design procedures simpler than the original procedures of Luenberger (1964, 1966). Bongiorno and Youla (1968) proposed a new observer design technique and showed that if the estimate of a system's state provided by an observer is to replace the state in an optimal (with respect to a quadratic performance index) control law, then proper care must be exercised in selecting the location of the observer poles in order to avoid excessive degradation in the value of the optimal cost functional (see also Bongiorno and Youla, 1970). Recently, Bongiorno (1973) considered the related problem of observer design for insensitivity of a quadratic performance index to variations in the observed system parameters, and showed that complete insensitivity is, in general, unattainable. Wonham (1970) studied the properties of observers from the geometric point of view and removed some existing restrictions on the allowable locations for the observer poles(that the observer and observed system cannot have any poles in common, and that some observer poles are allowed to assume real values exclusively).

Among the several observer design techniques that appeared in the next few years are those of Johnson (1969), Gopinath (1971), Jameson and Rothschild (1971), and Munro (1973, 1974). These procedures are all based on state variable descriptions of both the system and observer. Among these, notable for their simplicity are the (related) design techniques of Johnson (1969) and Gopinath (1971). A transfer function matrix description observer design method has been proposed by Retallack (1970), and Wolovich (1973a) has used the matrix-fraction description approach to design observers.

When the literature became saturated with design procedures to observe the complete state, researchers turned naturally to the more difficult problems of designing observers to approximate scalar and vector linear functions of the state with minimal and/or low order observers. For the scalar linear function case, the efforts of Jameson and Rothschild (1971), Wonham and Morse (1972), and Murdoch (1973) resulted in three different techniques to design an observer of dimension equal to $\nu - 1$ and with arbitrary poles. Anderson and Moore (1971) give an excellent discussion of the original solution proposed by Luenberger (1966) for this case. Williamson (1970), Wang and Davison (1973), and Roman et al. (1973) considered the problem of designing observers of dimension less than $\nu - 1$ and with restricted pole positions.

For the case of observing vector linear functions of the state a brute-force solution has been proposed by Jameson and Rothschild (1971). They suggest observing every row of the vector function with a $(\nu-1)$-dimensional observer and then combining the individual observers in parallel into a single $m(\nu-1)$-dimensional observer (where m is the number of rows of the given vector linear function). Wolovich (1973a)

formulated the observer design problem in terms of matrix-fraction descriptions, and obtained a result which is equivalent to that of Jameson and Rothschild (1971). It is noted that their solution is not minimal except in very unusual cases. Murdoch (1974) goes one step farther than Jameson and Rothschild (1971) and allows coupling in the forward direction between the individual observers. This reduces considerably the dimension of the overall observer, but is still a non-minimal solution in most cases.

Fortmann and Williamson (1972) proposed a method which consists of designing r minimal order subobservers to estimate r partitions of the given set of vector linear functions and then combining the subobservers in parallel to obtain the overall observer. This procedure is a dual of the solution proposed by Jameson and Rothschild (1971), and correspondingly gives non-minimal observers except in very special cases.

The strongest results available prior to this work for the problem of observing vector linear functions of the state are due to Wang and Davison (1973) and Roman and Bullock (1974). Wang and Davison (1973) reformulate the problem in transfer function matrix terms as follows: given two rational matrices $\Omega(s)$ and $\Delta(s)$ (the transposes of the system transfer function matrix and the desired overall transfer function matrix, respectively), find a rational matrix $\Lambda(s)$ (the transpose of the observer transfer function matrix) of lowest possible McMillan degree (see McMillan, 1952; also, Rosenbrock, 1970) such that

(1.1) $$\Omega(s)\Delta(s) = \Lambda(s)$$

Substituting the matrix-fraction description for each of the transfer functions in (1.1) and performing some straightforward manipulations, they represent (1.1) in a form in which it is possible to solve for the

minimal $\Lambda(s)$ that satisfies this relation. However, there is no guarantee that the poles are located so that $\Lambda(s)$ will be stable. If $\Lambda(s)$ is unstable, it does not qualify for the transfer function of an observer. Thus, Wang and Davison (1973) cannot design a minimal order observer in every case, and further, the success or failure of the technique is not known until $\Lambda(s)$ has been computed.

The results described by Roman and Bullock (1974) are a preliminary version of the observer design procedures proposed in this dissertation, and are amply discussed in the sequel. Several of the above-mentioned observer design techniques are also discussed in Chapter 4 and compared with the methods presented here. Specifically, the observer design techniques reported by Luenberger (1966), Gopinath (1971), Munro (1973,1974), Fortmann and Williamson (1972), and Murdoch (1974) are examined in more detail.

Several extensions to the original observer design problem have been formulated and discussed in the last few years. Wolovich (1968), Johnson (1969), Williamson (1970), Yüksel and Bongiorno (1971), and Watts and McDaniel (1973) have considered the design of observers for time-varying systems, and Stavroulakis and Sarachik (1973) have studied the design of observers for distributed parameter systems.

Haley (1967) and Russell and Bullock (1974) considered the design of low order observers for square systems (systems having equal number of inputs and outputs) using approximation techniques.

Guidorzi and Marro (1971), Hostetter and Meditch (1973), and Meditch and Hostetter (1973) discuss the problem of designing observers for systems with unmeasurable inputs.

The design of observers for nonlinear systems has been discussed by Williamson (1970), Thau (1973), and Kou et al. (1973) from three different points of view.

The so-called observer-estimators, which are observers for stochastic systems which have some noise-free measurements, were introduced by Tse and Athans (1970). These are becoming increasingly popular due to their practical applications. Among the various researchers that have since considered this problem are Iglehart and Leondes (1972), Tse (1973a) Leondes and Novak (1974), Kwatny (1974), and Uttam and O'Halloran (1975).

Carroll and Lindorff (1973, 1974) originated the adaptive observer trend, and Lüders and Narendra (1973, 1974a, b), and Tse (1973b) have followed closely. More recently, Morse (1974) has studied the design of adaptive observers using the powerful algebraic tools developed by Morse and Wonham (1970).

All indications seem to be that future research will be mostly in the design of observer-estimators, adaptive observers, and observers for nonlinear systems. However, most of the other areas mentioned above have not been completely exhausted, and still offer considerable challenge.

Statement of Purpose and Chapter Outline

It is the purpose of this disseration to provide an extensive discussion of realization theory and present complete and partial realization algorithms which are superior to other existing algorithms. Also, two new partial realization problems of interest in applications are defined. They are the minimal partial stable and the minimal partial arbitrary realization problems. Solutions for these problems are

presented. Then the previously unsolved problems of designing minimal order observers and minimal order observers with arbitrary poles to estimate vector linear functions of the state are considered. It is shown that these problems can be formulated as partial realization problems, and the partial realization algorithms given here are applied to obtain a solution which has several significant features. In passing, a considerable amount of knowledge about the existence and structure of observers is gained.

An outline by chapters follows. The invariant structure of linear, constant systems and the minimal realization of infinite and finite impulse response matrix sequences are discussed in Chapter 2. Two realization algorithms (one complete and the other partial) are presented.

The cases where a stable realization or a realization with arbitrary poles must be obtained for a finite matrix sequence are considered and solved in Chapter 3 through an extension of the concepts and algorithms given in Chapter 2.

In Chapter 4 the design of minimal order observers, minimal order observers with arbitrary poles, and intermediate order observers (to estimate vector linear functions of the state) is formulated in terms of partial realization theory and the results of the preceding chapters are applied to the specific context of observers.

The specific contributions of this research and further research possibilities are outlined in Chapter 5.

Examples are used generously throughout this work to illustrate the various realization algorithms discussed and to point out significant details that are otherwise difficult to see. A comment on the notation to be used throughout this dissertation closes this chapter.

(1.2) Notation. Uppercase letters denote matrices, and vectors are represented by underlined lowercase letters. Lowercase letters are used to represent scalars and integers. Sets and their elements are denoted by upper- and lowercase script letters, respectively.

Lowercase Greek symbols are used (with a few exceptions) to denote the invariants of the group of similarity transformations on the state-space of linear systems, and occasionally, to represent polynomials. Uppercase Greek symbols stand for polynomial matrices or rational matrices.

Several auxiliary matrices and vectors are represented by upper- and lowercase, respectively, light italic type elements. The symbol $\mathcal{H}$ stands for the so-called generalized Hankel matrix. All the matrices and vectors appearing in this work are assumed to be real and constant. I_n represents the n×n identity matrix, and $\underline{i}_j$ stands for its jth column. A p×q null matrix is denoted by $0_{p,q}$, and $\underline{0}_p$ is one of its columns. ∇

CHAPTER 2
COMPLETE AND PARTIAL REALIZATION THEORY

In this chapter the problems of realizing a) an infinite, and b) a finite (or partial) matrix sequence with a system of minimal order are formulated and discussed. A complete system of input invariants of the complete and partial realization problems is obtained, and a state-space canonical description is defined. It is shown that the columns of the so-called Hankel array and the columns of the matrices in the given sequence satisfy a set of recurrence relations defined by the system of input invariants of the realization. Based on these results, realization algorithms are presented to solve the above-mentioned problems. These algorithms have the significant feature that the realization is obtained in canonical form, and further, the complete realization algorithm is of quasi-recursive nature.

The theory of complete and partial realizations has been discussed extensively in the control literature since the pioneering work of Gilbert (1963), Kalman (1963), Zadeh and Desoer (1963), and Ho and Kalman (1966). The contribution of this chapter lies in the description of new realization algorithms. Results similar to those described here have appeared recently in the complete realization context (Bonivento et al., 1973; Rissanen, 1974; Akaike, 1974), but several significant points which are discussed here seem to have been overlooked elsewhere. Further, it should be noted that the complete realization algorithm given here was derived independently of these papers.

Problem Statement

Consider a system of the form[†]

(2.1a) $$\dot{\underline{\lambda}} = A\underline{\lambda} + Q\underline{\omega}$$

(2.1b) $$\underline{\eta} = D\underline{\lambda}$$

where $\underline{\lambda}$ is the p-dimensional state vector, $\underline{\omega}$ is the r-dimensional input vector, $\underline{\eta}$ is the m-dimensional output vector, and A, Q, D are matrices of appropriate dimensions. System (2.1) is also denoted by the matrix triple $(A,Q,D)_p$. Without loss of generality, it is assumed that (2.1) defines a completely controllable and observable system. For notational simplicity, let $\rho(Q) = r$, $\rho(D) = m$.

The transfer function matrix of (2.1) is given by

(2.2) $$\Phi(s) = D[sI - A]^{-1}Q = \sum_{i=1}^{\infty} L_i s^{-i}$$

and the impulse response matrix of (2.1) is given by

(2.3) $$L(t) = De^{At}Q = D[\sum_{i=1}^{\infty} \frac{A^{i-1}t^{i-1}}{(i-1)!}]Q = \sum_{i=1}^{\infty} \frac{t^{i-1}}{(i-1)!} L_i$$

Thus, an alternate description of (2.1) consists of the infinite matrix sequence $\{L_1,L_2,...\}$. This leads to the following definition of the minimal complete realization problem.

(2.4) Definition. Given an infinite sequence $\{L_1,L_2,...\}$ of m×r matrices which define the impulse response of a finite-dimensional system (2.1), find a triple $(A,Q,D)_p$ such that

[†] The notation used in this chapter differs considerably from the standard notation. This discrepancy is introduced in the interest of simplifying the notation in a subsequent chapter.

a) $L_i = DA^{i-1}Q$ $\qquad i = 1,2,\ldots$

b) p is minimal.

The triple $(A,Q,D)_p$ is then said to be a minimal complete realization of $\{L_1,L_2,\ldots\}$.

In a practical situation, the input/output data is available only partially. Thus, a more realistic problem is the minimal partial realization problem, which is defined as follows.

(2.5) Definition. Given a finite sequence $\{L_1,L_2,\ldots,L_{N_0}\}$ of $m\times r$ matrices which define the first N_0 terms in the impulse response of a (possibly infinite-dimensional) system, find a triple $(A,Q,D)_p$ such that

a) $L_i = DA^{i-1}Q$ $\qquad i = 1,2,\ldots,N_0$

b) p is minimal.

The triple $(A,Q,D)_p$ is then said to be a minimal partial realization of $\{L_1,L_2,\ldots,L_{N_0}\}$

(2.6) Remark. For a continuous-time system, as (2.1), the low-index terms of the sequence $\{L_1,L_2,\ldots\}$ define the high-frequency response of (2.1). Thus, realizing the first N_0 terms of the sequence $\{L_1,L_2,\ldots\}$ may result (depending on the size of N_0) in a system which does not give a satisfactory low-frequency (steady-state) response. It is well known (Ho and Kalman, 1966) that for discrete-time systems the realization problem can be defined in the exact same form as above. However, in the discrete-time case the value of the impulse response at the ith sampling instant is given by L_i. Thus, a minimal partial realization of the impulse response sequence of a discrete-time system matches exactly the desired input/output behavior up to the N_0th term. Most of

the results of this dissertation are equally applicable to discrete-time systems, but are developed here in the context of continuous-time systems only.

Two interesting and important variants of the partial realization problem are defined and discussed in the next chapter. These are the minimal partial stable realization and the minimal partial arbitrary realization problems.

The approach taken in the next sections is to analyze the basic properties of all minimal factorizations of the form (2.4a) and (2.5a) for infinite and finite matrix sequences, respectively, and use such information to formulate simple realization algorithms. The ideas and concepts developed in this chapter also extend to the problems discussed in Chapter 3.

Some Definitions and Known Results

Several definitions and well-known results which provide a basis for the main results of this chapter are stated below.

(2.7) Definition. The generalized Hankel matrix $\mathcal{H}$ of an infinite sequence $\{L_1, L_2, ...\}$ is defined as the following infinite-dimensional matrix

$$\mathcal{H} = \begin{bmatrix} L_1 & L_2 & L_3 & . & . & . \\ L_2 & L_3 & L_4 & . & . & . \\ L_3 & L_4 & L_5 & . & . & . \\ . & . & . & & & \\ . & . & . & & & \\ . & . & . & & & \end{bmatrix}$$

and the jth column of the kth block column of $\mathcal{H}$ is denoted by $\underline{h}_{jk}$. Also, $\underline{\ell}_{jk}$ is defined to be the jth column of L_k.

(2.8) Definition. The generalized truncated Hankel matrix $\mathcal{H}_{NN'}$ of an infinite sequence $\{L_1,L_2,...\}$ is defined as the following $mN\times rN'$ matrix

$$\mathcal{H}_{NN'} = \begin{bmatrix} L_1 & L_2 & \cdot & \cdot & \cdot & L_{N'-1} & L_{N'} \\ L_2 & L_3 & \cdot & \cdot & \cdot & L_{N'} & L_{N'+1} \\ \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot \\ L_{N-1} & L_N & \cdot & \cdot & \cdot & L_{N+N'-3} & L_{N+N'-2} \\ L_N & L_{N+1} & \cdot & \cdot & \cdot & L_{N+N'-2} & L_{N+N'-1} \end{bmatrix}$$

and its $(r(k-1)+j)$th column is also denoted by $\underline{h}_{jk}$. Whether $\underline{h}_{jk}$ is an infinite vector or not will be made clear from the context.

The following fundamental result of realization theory is stated without proof (see Kalman et al., 1969, for the proof).

(2.9) Theorem. (Realizability Criterion) Consider an infinite sequence $\{L_1,L_2,...\}$ of $m\times r$ matrices which admits a finite-dimensional realization. System (2.1) is a minimal realization of the given sequence if and only if

(2.10) $$\rho(\mathcal{H}_{NN'}) = \rho(\mathcal{H}_{N+iN'+j}) \qquad \begin{matrix} i = 0,1,2,... \\ j = 0,1,2,... \end{matrix}$$

for some integers N and N'. The dimension of the minimal realization is given by $p_M = \rho(\mathcal{H}_{NN'})$.

This theorem implies that the columns (rows) of $\mathcal{H}$ after the rN'th (mNth) column (row) are linearly dependent upon the preceding columns (rows). It is shown below that this linear dependency occurs in a recursive fashion (that is, the coefficients by which the $(rN'+j)$th column for $j = 1,2,...,r$, depends on its preceding rN' columns are the

same coefficients by which the $(rN'+j+k)$th column for $k = 1,2,\ldots,$ depends on its preceding rN' columns). Generally, these linear dependencies can be defined by more than one set of coefficients (depending on which specific basis is chosen), but among all the possible sets there is one that has a direct relation with the basic input (output) structure of a minimal realization (2.1) of the sequence. It is this the set of coefficients which is sought here.

Given a system of the form (2.1), its impulse response sequence can be generated using (2.2) or (2.3). In the minimal partial realization problem it is required that the first N_0 terms of the impulse response sequence of a realization match exactly the specified partial sequence. This suggests the following solution. Append to the given partial sequence an extension sequence $\{L_{N_0+1}, L_{N_0+2}, \ldots\}$ and then solve the realization problem using any one of several realization schemes (for example,Ho and Kalman, 1966; Rissanen, 1971; Ackermann and Bucy, 1971; Silverman, 1971). However, one has to be cautious in selecting an extension sequence. A poorly chosen extension sequence can lead to a high-dimension (possibly infinite) realization.

Consider a partial sequence $\{L_1, L_2, \ldots, L_{N_0}\}$ of $m \times r$ matrices, and form the associated Hankel matrix $\mathcal{H}$ as follows.

$$
(2.11) \qquad \mathcal{H} = \begin{bmatrix}
L_1 & L_2 & \cdots & L_{N_0-1} & L_{N_0} & * & * & \cdots \\
L_2 & L_3 & \cdots & L_{N_0} & * & * & * & \cdots \\
 & & & \vdots & & & & \\
L_{N_0-1} & L_{N_0} & * & * & \cdots & * & * & \cdots \\
L_{N_0} & * & * & * & \cdots & * & * & \cdots \\
* & * & * & * & \cdots & * & * & \cdots \\
 & & & \vdots & & & &
\end{bmatrix}
$$

where the asterisks (*) represent the (as yet unspecified)elements of an extension sequence, and $\mathcal{H}_{N_0 N_0}$ is the $mN_0 \times rN_0$ upper left-hand block of $\mathcal{H}$. Suppose that the unspecified elements in (2.11) are chosen such that a column (row) of $\mathcal{H}$ which is linearly dependent upon its preceding columns (rows) before the specification of the extension sequence, remains linearly dependent upon its preceding columns (rows) after the specification of the extension sequence. It is clear that replacing the asterisks in (2.11) with such an extension sequence cannot increase the rank of $\mathcal{H}$. Then, if such a selection of the unspecified elements can indeed be made, it follows from Theorem (2.9) that

$$
(2.12) \qquad p_M = \rho(\mathcal{H}) = \rho(\mathcal{H}_{N_0 N_0}) = p_1 + p_2 + \ldots + p_{N_0}
$$

where

p_1 = number of linearly independent columns in the block column

$$
\begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_{N_0} \end{bmatrix}
$$

p_2 = number of linearly independent columns in the block column

$$\begin{bmatrix} L_2 \\ L_3 \\ \cdot \\ \cdot \\ \cdot \\ L_{N_0} \end{bmatrix}$$

that are also linearly independent of the columns in the block column

$$\begin{bmatrix} L_1 \\ L_2 \\ \cdot \\ \cdot \\ \cdot \\ L_{N_0-1} \end{bmatrix}$$

$\vdots$

p_{N_0} = number of linearly independent columns in the matrix L_{N_0} that are also linearly independent of the columns of the matrices $L_1, L_2, \ldots, L_{N_0-1}$.

Kalman (1971b) and Tether (1970) have independently shown that an extension sequence always exists for which the dimension of the realization is given by p_M, and that (generally) it is not unique. An extension sequence which corresponds to a minimal realization (that is, a realization of dimension p_M) is said to be a minimal extension sequence. This result is now stated as a theorem.

(2.13) Theorem. Given a finite sequence $\{L_1, L_2, \ldots, L_{N_0}\}$ of $m \times r$ matrices, the dimension of its minimal realization is given by (2.12).

The proof of Theorem (2.13) appears in the work of Kalman (1971b) and of Tether (1970), and is not repeated here. It consists of a systematic procedure for selecting the columns (rows) of the matrices in the extension sequence so that existing linear dependencies in $\mathcal{H}$ are not disturbed. In the process, it becomes apparent that there will generally be some arbitrariness involved in the selection of an extension sequence which does not increase the rank of $\mathcal{H}$ from the value p_M. That is, the minimal extension sequence is generally not unique. This nonuniqueness can conceivably be used to obtain a minimal realization with some desirable characteristics (for example: cyclicity, as many stable poles as possible, and others). Unfortunately, the construction used by Kalman (1971b) and Tether (1970) does not give much insight for doing so because the number and nature of the allowable degrees of freedom is not easily recognized (except in the scalar case). Godbole (1972) has pointed out some of these problems via examples.

Consider now the following remarks.

(2.14) Remark. It is clear that the integers N, N', p_M, p_1, ..., p_{N_0} are non-decreasing functions of N_0, and should be written as $N(N_0)$, $N'(N_0)$, $p_M(N_0)$, etc. However, to maintain notational simplicity the argument N_0 is not used in this dissertation. ∇

(2.15) Remark. Equation (2.12) can be written also from the dual point of view. That is, the linearly independent rows of $\mathcal{H}_{N_0 N_0}$ can be considered instead of its columns, and the resulting p_M is the same value as that calculated using (2.12). ∇

(2.16) Remark. In the complete realization context, a minimal realization is both completely controllable and observable (otherwise, pole-zero cancellation in the transfer function matrix would reduce the

dimension of the realization), and vice versa, a completely controllable and observable realization of an infinite matrix sequence is also minimal (this is a consequence of the uniqueness, modulo a basis change in the state-space, of complete realizations). However, while it is still true in the partial realization context that minimality implies complete controllability and observability, it is not always true that a completely controllable and observable realization of a partial sequence is also minimal. This is a consequence of the nonuniqueness of partial realizations. A more detailed discussion of the nonuniqueness of partial realizations is given at the end of this chapter and in the next chapter. ∇

In all the above discussions and in the above remark, the consideration of infinite and finite sequences all of whose elements are zero (referred to as null or zero sequences) has been implicitly ruled out. This is so because the dimension of the minimal realization of a null sequence (both partial and complete) is equal to zero, and such a realization is not of the form (2.1).

Non-minimal realizations of an infinite null sequence are trivially given by any completely uncontrollable and/or unobservable system of the form (2.1), but such realizations are not very meaningful. In contrast, it is possible to construct completely controllable and observable non-minimal realizations for a finite null sequence, and it can be shown (based on the theory presented in Chapter 3) that such realizations have arbitrary structure and poles. But in the application of partial realization theory to observer design (which is one of the main objectives of this dissertation) the occurrence of a partial null sequence has a trivial interpretation. For these reasons, it is assumed that all sequences (whether partial or complete) considered in this dissertation are non-zero unless stated otherwise.

Realization Invariants and a Canonical Form

The basic properties common to all minimal factorizations of the form (2.4a) and (2.5a) for infinite and finite matrix sequences, respectively, are studied next. This leads to the definition of a complete system of invariants for minimal complete and partial realizations, and the construction of a set of state-space canonical forms. The matrix-fraction description of system (2.1) (which is becoming quite popular in the current literature) is also considered briefly, and a relation between the set of state-space canonical forms and a corresponding matrix-fraction description is presented. These results are the basis for the minimal realization algorithms of the next section and Chapter 3.

Consider first the following definitions.

(2.17) Definition. The controllability index of a completely controllable matrix pair $(A,Q)_p$ is defined as the smallest positive integer μ such that the matrix

$$\hat{C} = [Q \quad AQ \quad . \; . \; . \quad A^{\mu-1}Q]$$

has rank p.

(2.18) Definition. The observability index of a completely observable matrix pair $(A,D)_p$ is defined as the smallest positive integer δ such that the matrix

$$\hat{O} = \begin{bmatrix} D \\ DA \\ \cdot \\ \cdot \\ \cdot \\ DA^{\delta-1} \end{bmatrix}$$

has rank p.

(2.19) Definition. Let q_i denote the ith column of Q. Select the first p linearly independent columns of $\hat{C}$ and rearrange them as follows

(2.20)
$$q_1, Aq_1, \ldots, A^{\mu_1 - 1} q_1, q_2, \ldots, A^{\mu_2 - 1} q_2, \ldots, q_r, \ldots, A^{\mu_r - 1} q_r$$

The integer μ_i which denotes the number of columns of $\hat{C}$ in the above basis that are associated with q_i is called the ith controllability index of the matrix pair $(A,Q)_p$, and the set of integers $\{\mu_i\}$ is called the set of controllability indices of the pair $(A,Q)_p$. The dual set of observability indices $\{\delta_i\}$ of the matrix pair $(A,D)_p$ is defined analogously by considering the first p linearly independent rows of $\hat{O}$. It is clear that the $\{\mu_i\}$ and $\{\delta_i\}$ are well-defined, unique sets of positive (given $\rho(Q) = r$ and $\rho(D) = m$) integers, and that

$$p = \mu_1 + \mu_2 + \ldots + \mu_r = \delta_1 + \delta_2 + \ldots + \delta_m$$
$$\mu = \max(\mu_i)$$
$$\delta = \max(\delta_i)$$

By a slight abuse of notation, the $\{\mu_i\}$ and $\{\delta_i\}$ are also called the controllability (or input) and observability (or output) indices, respectively, of system (2.1).

The integers $\{\mu_i\}$ are also referred to as the Kronecker or minimal indices of a matrix pair $(A,Q)_p$ because of their relation to the minimal indices of the matrix pencil $[sI - A \quad Q]$. A dual remark holds for the integers $\{\delta_i\}$. This relation has been discussed by Rosenbrock (1970) and Kalman (1971a).

The following lemma is basic to all the developments of this chapter. It seems to have appeared first in a paper by Rissanen (1974), and more recently, it appears mentioned in the work of Denham (1974), Dickinson et al. (1974a, b), and Akaike (1974). Roman and Bullock (1975) have derived it independently, and that approach is the one followed here.

(2.21) Lemma. Consider an infinite sequence $\{L_1, L_2, \ldots\}$ of $m \times r$ matrices which admits a finite-dimensional minimal realization (2.1), and let $\mathcal{H}$ be the associated Hankel matrix. Then the columns of $\mathcal{H}$ satisfy the following recurrence relations

(2.22)
$$\underline{h}_{i\mu_i+1+\tau} = -\sum_{j=1}^{r} \sum_{k=1}^{\sigma_{ij}} \gamma_{ijk} \underline{h}_{jk+\tau} \qquad i = 1,2,\ldots,r \quad \tau = 0,1,2,\ldots$$

where the $\{\gamma_{ijk}\}$ are real scalars, the $\{\mu_i\}$ are the controllability indices of the matrix pair $(A,Q)_p$, and the $\{\sigma_{ij}\}$ are given by

(2.23)
$$\sigma_{ij} = \begin{cases} \mu_i + 1 & \text{for } \mu_i < \mu_j,\ i > j \\ \mu_i & \text{for } \mu_i \leq \mu_j,\ i \leq j \\ \mu_j & \text{for } \mu_i \geq \mu_j,\ \forall i, j \end{cases}$$

Proof. Define C and O as the following infinite matrices

(2.24)
$$C = [Q \quad AQ \quad A^2Q \ . \ . \ .]$$

(2.25)
$$O = \begin{bmatrix} D \\ DA \\ DA^2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Then matrices $\hat{C}$ and $\hat{O}$ defined previously are $p \times \mu p$ and $\delta p \times p$ submatrices of C and O, respectively.

System (2.1) is a minimal realization of the given sequence, so $(A,Q)_p$ and $(A,D)_p$ are a completely controllable and observable pair, respectively. Thus, there are exactly p linearly independent columns in C and in O. Select a basis for the p-dimensional space from the columns of C using the selection procedure of Definition (2.19). This identifies the controllability indices of $(A,Q)_p$.

From the procedure used to select the basis (2.20), it follows that $A^{\mu_i}\underline{q}_i$, for $1 \leq i \leq r$, can be expressed as a linear combination of those vectors in (2.20) which precede it in C. More specifically, $A^{\mu_i}\underline{q}_i$ is a linear combination of the vectors

$$
(2.26)\qquad
\begin{array}{ll}
\underline{q}_i,\ A\underline{q}_i,\ \ldots,\ A^{\mu_i-1}\underline{q}_i & \\
\underline{q}_j,\ A\underline{q}_j,\ \ldots,\ A^{\mu_i-1}\underline{q}_j & \text{for } \mu_i \leq \mu_j \text{ and } i < j \\
\underline{q}_j,\ A\underline{q}_j,\ \ldots,\ A^{\mu_i}\underline{q}_j & \text{for } \mu_i < \mu_j \text{ and } i > j \\
\underline{q}_j,\ A\underline{q}_j,\ \ldots,\ A^{\mu_j-1}\underline{q}_j & \text{for } \mu_i \geq \mu_j \text{ and } i \neq j
\end{array}
$$

where both i and j range from 1 to r. Notice that the exponent of the largest power of A in (2.26) is given by $\sigma_{ij} - 1$, with σ_{ij} defined as in (2.23).

The dependency of $A^{\mu_i}\underline{q}_i$ on the vectors listed in (2.26) can be expressed in equation form as

$$
(2.27)\qquad A^{\mu_i}\underline{q}_i = -\sum_{j=1}^{r}\sum_{k=1}^{\sigma_{ij}} \gamma_{ijk}A^{k-1}\underline{q}_j \qquad i = 1,2,\ldots,r
$$

where the $\{\gamma_{ijk}\}$ are well-defined, unique, real scalars, and the $\{\sigma_{ij}\}$ are as in (2.23). From (2.27) it follows that

(2.28) $$A^{\mu_i+\tau} \underline{q}_i = - \sum_{j=1}^{r} \sum_{k=1}^{\sigma_{ij}} \gamma_{ijk} A^{k-1+\tau} \underline{q}_j \qquad i = 1,2,\ldots,r \quad \tau = 0,1,2,\ldots$$

In words, (2.28) states that the columns of C satisfy a set of r recurrence relations with coefficients $\{\gamma_{ijk}\}$.

Since $\rho(C) = \rho(O) = \rho(\mathcal{H}) = p$, it follows that the vectors

(2.29) $$O\underline{q}_1, OA\underline{q}_1, \ldots, OA^{\mu_1-1}\underline{q}_1, O\underline{q}_2, \ldots, OA^{\mu_2-1}\underline{q}_2, \ldots, O\underline{q}_r, \ldots, OA^{\mu_r-1}\underline{q}_r$$

constitute a basis for the n-dimensional space; further,

(2.30) $$OA^{\mu_i+\tau} \underline{q}_i = - \sum_{j=1}^{r} \sum_{k=1}^{\sigma_{ij}} \gamma_{ijk} OA^{k-1+\tau} \underline{q}_j \qquad i = 1,2,\ldots,r \quad \tau = 0,1,2,\ldots$$

Finally, (2.22) follows from (2.30) and the definition of $\underline{h}_{jk}$ (see Definition (2.7)).

∇

(2.31) Corollary. The columns of the matrix elements in the infinite impulse response sequence $\{L_1, L_2, \ldots\}$ of system (2.1) satisfy the following recurrence relations

(2.32) $$\underline{\ell}_{i\mu_i+1+\tau} = - \sum_{j=1}^{r} \sum_{k=1}^{\sigma_{ij}} \gamma_{ijk} \underline{\ell}_{jk+\tau} \qquad i = 1,2,\ldots,r \quad \tau = 0,1,2,\ldots$$

where the $\{\mu_i\}$, $\{\gamma_{ijk}\}$, and the $\{\sigma_{ij}\}$ are as previously defined.

Proof. It follows from (2.30), the definition of $\underline{\ell}_{jk}$, and the Hankel pattern.

∇

(2.33) Remark. Lemma (2.21), Corollary (2.31), and most of the results that are given here can also be stated from the dual point of view. This is done by considering the rows of O and $\mathcal{H}$ instead of the columns of C and $\mathcal{H}$. For certain purposes (to be seen later on), the dual

approach proves to be more convenient. In the dual context, the counterparts of the parameters $\{\gamma_{ijk}\}$ and $\{\sigma_{ij}\}$ are denoted by the sets $\{\theta_{ijk}\}$ and $\{\kappa_{ij}\}$, respectively.

A close examination of Corollary (2.31) suggests that the parameters $\{\mu_i\}$, $\{\gamma_{ijk}\}$, and the vectors $\{\underline{\ell}_{jk}\}$ for $i, j = 1,2,\ldots,r$ and $k = 1, 2,\ldots,\sigma_{ij}$, constitute all the information necessary to completely specify an infinite sequence $\{L_1,L_2,\ldots\}$ which admits a finite-dimensional realization of dimension p. This point can be discussed more precisely after the introduction of several standard definitions (see, for example, MacLane and Birkhoff, 1967).

(2.34) Definition. A group is a set of elements, denoted by G, which is closed under a binary operation, denoted by $\square$, such that

a) the operation $\square$ is associative:

$$x\square(y\square z) = (x\square y)\square z \qquad \forall\ x,y,z \in G$$

b) G contains an identity element e:

$$x\square e = e\square x = x \qquad \forall\ x \in G$$

c) G contains an inverse for every element:

$$x\square x^{-1} = x^{-1}\square x = e \qquad \forall\ x \in G$$

The group $(G,\square)$ is denoted by G alone when the associated operation $\square$ is understood.

(2.35) Definition. A binary relation, denoted by $\sim$, on a set S is said to be an equivalence relation if it has the properties of

a) reflexivity: $x \sim x \qquad \forall\ x \in S$

b) symmetry: $x \sim y$ implies $y \sim x\ \forall\ x,y \in S$

c) transitivity: $x \sim y$ and $y \sim z$ imply $x \sim y$

$$\forall\ x,y,z \in S$$

The set $E(x) = \{y : y \in S \text{ and } y \sim x\}$ is called the equivalence class of x.

(2.36) Definition. A function $f : S \to W$, where S and W are sets, is an invariant for the equivalence relation $\sim$ if $x \sim y$ implies $f(x) = f(y)$ for $x, y \in S$. The function f is a complete invariant for $\sim$ if $x \sim y$ implies $f(x) = f(y)$, and $f(x) = f(y)$ implies $x \sim y$. A list $f_1, f_2, \ldots, f_j$ of functions $f_i : S \to W_i$ is a complete system of invariants for $\sim$ when each f_i is an invariant for $\sim$, and $f_1(x) = f_1(y)$, $f_2(x) = f_2(y)$, ..., $f_j(x) = f_j(y)$ imply $x \sim y$.

(2.37) Definition. A subset D of a set S is a set of canonical forms for the equivalence relation $\sim$ if to each subset $E(x)$ of S there corresponds exactly one element d in D; this element d is then the canonical form of every element y in $E(x)$.

The following lemma can be stated. The proof is simple, and is not given; it consists of verifying the postulates of Definition (2.34).

(2.38) Lemma. The set all pxp nonsingular matrices, denoted by T, together with the matrix multiplication operation, denoted by $\cdot$, forms a group.

(2.39) Definition. The group $(T, \cdot)$ of Lemma (2.38) is called the general linear group, and is denoted as GL(p). The set of all controllable and observable matrix triples $(A,Q,D)_p$ is denoted by $S(p)$.

The set $S(p)$ and the group GL(p) are the entities of interest in this dissertation. An equivalence relation is defined next for the elements of $S(p)$ under the action of GL(p). Then a complete system of

invariants is obtained for that equivalence relation, and a subset of $S(p)$ of canonical forms is defined. This leads to the formulation of simple realization algorithms to solve the realization problems previously defined. The following simple lemmas are useful.

(2.40) Lemma. A change of basis on the state-space of (2.1) corresponds to an equivalence relation defined by GL(p) acting on $S(p)$ as follows:

(2.41)
$$(A_1,Q_1,D_1)_p \mapsto (TA_1T^{-1},TQ_1,D_1T^{-1})_p = (A_2,Q_2,D_2)_p$$

for any T in GL(p).

Proof. It is required that the postulates of Definition (2.35) be satisfied. Reflexivity is shown by considering $T = I_p$ in (2.41). Symmetry follows using T^{-1}:

$$(A_2,Q_2,D_2)_p \mapsto (T^{-1}A_2T,T^{-1}Q_2,D_2T)_p =$$

$$(T^{-1}TA_1T^{-1}T,T^{-1}TQ_1,D_1T^{-1}T)_p = (A_1,Q_1,D_1)_p$$

To show transitivity, let

$$(A_1,Q_1,D_1)_p \mapsto (T_1A_1T_1^{-1},T_1Q_1,D_1T_1^{-1})_p = (A_3,Q_3,D_3)_p$$

$$(A_3,Q_3,D_3)_p \mapsto (T_2A_3T_2^{-1},T_2Q_3,D_3T_2^{-1})_p = (A_2,Q_2,D_2)_p$$

and take $T = T_2T_1$ in (2.41). ∇

(2.42) Lemma. Two elements of $S(p)$ that are equivalent under the action of GL(p) have the same impulse response sequence.

Proof. Suppose $(A_1,Q_1,D_1)_p$ and $(A_2,Q_2,D_2)_p$ in $S(p)$ are equivalent. Let the impulse response sequence of $(A_1,Q_1,D_1)_p$ be given by

$$L_i = D_1A_1^{i-1}Q_1 \qquad i = 1,2,\ldots$$

Then it follows from (2.41) that

$$D_2A_2^{i-1}Q_2 = (D_1T^{-1})(TA_1T^{-1})^{i-1}(TQ_1) \qquad i = 1,2,\ldots$$

$$= D_1A_1^{i-1}Q_1 = L_i \qquad i = 1,2,\ldots$$

as required. ∇

(2.43) Lemma. Any two minimal realizations of an infinite matrix sequence $\{L_1, L_2, \ldots\}$ are equivalent under the action of GL(p).

Proof. Let $\mathcal{H}$ be the Hankel matrix associated with a specified infinite matrix sequence $\{L_1, L_2, \ldots\}$ which admits a minimal realization of dimension p, and let $(A_1, Q_1, D_1)_p$ and $(A_2, Q_2, D_2)_p$ be two such minimal realizations. The Hankel pattern implies that

$$(2.44) \qquad \mathcal{H} = \begin{bmatrix} D_i \\ D_i A_i \\ D_i A_i^2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} [\, Q_i \quad A_i Q_i \quad A_i^2 Q_i \quad . \; . \; . \,] = \mathcal{O}_i \mathcal{C}_i \qquad i = 1,2$$

Let $\hat{\mathcal{O}}_i$ be the mp×p top submatrix of $\mathcal{O}_i$, and $\hat{\mathcal{C}}_i$ be the p×rp left submatrix of $\mathcal{C}_i$. Since a minimal realization of the given sequence has dimension p, then $\rho(\mathcal{H}) = \rho(\mathcal{H}_{pp}) = p$. Both realizations (1 and 2) are minimal, so $\rho(\mathcal{O}_i) = \rho(\hat{\mathcal{O}}_i) = \rho(\mathcal{C}_i) = \rho(\hat{\mathcal{C}}_i) = p$. Further, $\hat{\mathcal{O}}_i^T \hat{\mathcal{O}}_i$ and $\hat{\mathcal{C}}_i \hat{\mathcal{C}}_i^T$ are nonsingular. From (2.44),

$$(2.45) \qquad \hat{\mathcal{O}}_2 \hat{\mathcal{C}}_2 = \hat{\mathcal{O}}_1 \hat{\mathcal{C}}_1$$

and using the fact that $\hat{\mathcal{O}}_2^T \hat{\mathcal{O}}_2$ is nonsingular,

$$(2.46) \qquad \hat{\mathcal{C}}_2 = (\hat{\mathcal{O}}_2^T \hat{\mathcal{O}}_2)^{-1} \hat{\mathcal{O}}_2^T \hat{\mathcal{O}}_1 \hat{\mathcal{C}}_1 = T \hat{\mathcal{C}}_1$$

where T is p×p. That T is nonsingular can be shown by applying Sylvester's rank inequality (see, for example, Gantmacher, 1959). Further manipulations on (2.46) give

$$(2.47) \qquad I = T \hat{\mathcal{C}}_1 \hat{\mathcal{C}}_2^T (\hat{\mathcal{C}}_2 \hat{\mathcal{C}}_2^T)^{-1} = T T^{-1}$$

From (2.44) and (2.46) it follows that $Q_2 = TQ_1$, and

$$|A_2Q_2 \; A_2^2Q_2 \; \ldots \; A_2^pQ_2| = |TA_1Q_1 \; TA_1^2Q_1 \; \ldots \; TA_1^pQ_1|$$

$$A_2\hat{C}_2 = TA_1\hat{C}_1$$

$$A_2 = TA_1\hat{C}_1\hat{C}_2^T(\hat{C}_2\hat{C}_2^T)^{-1} = TA_1T^{-1}$$

The "dual" of (2.46) can also be obtained from (2.45). This is

(2.48) $$\hat{O}_2 = \hat{O}_1\hat{C}_1\hat{C}_2^T(\hat{C}_2\hat{C}_2^T)^{-1} = \hat{O}_1T^{-1}$$

It follows that $D_2 = D_1T^{-1}$. ∇

The preceding developments led to an important result on invariants for GL(p) acting on $S(p)$. It was probably obtained first by Rosenbrock (1970) via a frequency domain approach. Rosenbrock's derivation consists of performing a succession of equivalence transformations on the system matrix, which is defined to be the following $(p+m)\times(p+r)$ polynomial matrix

$$\begin{bmatrix} sI - A & Q \\ D & 0_{m,r} \end{bmatrix}$$

in order to obtain a canonical representation. More recently, Rissanen (1974) obtained the same result via both the time domain (state-space) and frequency domain points of view.

(2.49) Theorem. The parameters $\{\mu_i\}$ and $\{\gamma_{ijk}\}$, and the vectors $\{\underline{\ell}_{jk}\}$ for $i,j = 1,2,\ldots,r$ and $k = 1,2,\ldots,\sigma_{ij}$, with σ_{ij} defined by (2.23), constitute a complete system of invariants for the equivalence relation defined by the action of GL(p) on $S(p)$.†

†Common usage refers to the images of invariants as invariants themselves, but it should be kept in mind that invariants are functions, not concrete values (see Definition (2.36) and Remark (2.52)).

Proof. First it has to be shown that two elements of $S(p)$ which are equivalent under the action of $GL(p)$ have the same sets of $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ parameters and $\{\underline{\ell}_{jk}\}$ vectors. Then it must be proved that two elements of $S(p)$ which have the same sets of the above-mentioned invariants are related to each other as in (2.41).

Consider two equivalent elements $(A_1,Q_1,D_1)_p$ and $(A_2,Q_2,D_2)_p$ of $S(p)$. Lemma (2.42) states that they have the same impulse response sequence $\{L_1,L_2,\ldots\}$. Since the vectors $\{\underline{\ell}_{jk}\}$ are columns of the impulse response matrices, it follows that they are indeed invariants for $GL(p)$ acting on $S(p)$.

Given the Hankel matrix $\mathcal{H}$ of an infinite matrix sequence which admits a realization of dimension p, the integers $\{\mu_i\}$ are defined by the indices associated with the basis formed by the <u>first</u> p linearly independent columns of $\mathcal{H}$ (the basis (2.29)). This identifies the $\{\mu_i\}$ uniquely. The real scalars $\{\gamma_{ijk}\}$ are defined by equations (2.30). Since the representation of a vector in terms of a basis is unique, it follows that the $\{\gamma_{ijk}\}$ are unique also. Thus, the parameters $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ are invariants for $GL(p)$ acting on $S(p)$.

To prove the second part, suppose now that two elements $(A_1,Q_1,D_1)_p$ and $(A_2,Q_2,D_2)_p$ of $S(p)$ have the same sets of the above-mentioned invariants. The impulse response sequences of $(A_1,Q_1,D_1)_p$ and $(A_2,Q_2,D_2)_p$ can be constructed using (2.32). Obviously, the infinite sequences will be equal to each other. It follows from Lemma (2.43) that $(A_1,Q_1,D_1)_p$ and $(A_2,Q_2,D_2)_p$ are equivalent under the action of $GL(p)$. ∇

Popov (1972) has discussed the invariant problem for the set of all matrix pairs $(A,Q)_p$ (which represent system defined by (2.1a) only) under the action of $GL(p)$, and showed that the $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ are the

corresponding invariants. This result is also contained (indirectly)in the work of Kalman (1971a).

(2.50) Remark. The number of the $\{\gamma_{ijk}\}$ invariants depends on the value and ordering of the $\{\mu_i\}$ invariants. Given a specified set of the $\{\mu_i\}$, the number of $\{\gamma_{ijk}\}$ invariants associated with the same matrix triple is obtained as

$$(2.51) \qquad \sigma = \sum_{i=1}^{r} \sigma_i = \sum_{i=1}^{r} \sum_{j=1}^{r} \sigma_{ij}$$

where the $\{\sigma_{ij}\}$ are defined by (2.23). Based on the procedure used to compute the $\{\mu_i\}$, and examining (2.22), (2.27), or (2.32), it can be shown that $\sigma \leqq rp$. Further, it can also be shown (see Popov, 1972) that equality occurs if and only if $\mu_1 = \mu_2 = \ldots = \mu_r$, or if and only if $\mu_1 \geqq \mu_2 \geqq \ldots \geqq \mu_r > 0$ and $\mu_1 = \mu_2 = \ldots = \mu_j$, $\mu_{j+1} = \mu_{j+2} = \ldots = \mu_r = \mu_1 - 1$ for some $0 < j \leqq r - 1$. ∇

(2.52) Remark. Let $\mathcal{W}_1$ be the set of all r-dimensional vectors, $\mathcal{W}_2$ be the set of all σ-dimensional vectors, and $\mathcal{W}_3$ be the set of all mp-dimensional vectors. Then, in terms of Definition (2.36), $f_1 : \mathcal{S}(p) \to \mathcal{W}_1$ is the function which specifies the $\{\mu_i\}$ for a given matrix triple, $f_2 : \mathcal{S}(p) \to \mathcal{W}_2$ is the function which specifies the $\{\gamma_{ijk}\}$ for a given matrix triple, and $f_3 : \mathcal{S}(p) \to \mathcal{W}_3$ is the function which specifies the $\{\underline{\ell}_{jk}\}$ for a given matrix triple. The closed form of these functions is not necessary for the purposes of this dissertation, and from the procedures used to compute these sets of invariants, it seems that f_1, f_2, and f_3 are not simple functions of the elements of a matrix triple. The first part of the proof of Theorem (2.49) consisted of demonstrating that these functions (f_1,f_2,and f_3) are, individually, invariants for the action of GL(p) on $\mathcal{S}(p)$. However, they are not complete invariants if considered individually. ∇

(2.53) Remark. The integer p, the integer σ defined in (2.51), and the set of integers $\{\sigma_{ij}\}$ are also invariants for GL(p) acting on $S(p)$. This is also true of the dual of these and of the invariants of Theorem (2.49). But for the purposes of this dissertation, it seems to be more convenient to use the system of invariants of Theorem (2.49), or (in some cases) their dual counterparts. ∇

The theorem stated below gives a canonical state-space representation for the equivalence relation defined by a change of basis in the state-space of system (2.1). A canonical form for a matrix triple can be thought of as a "standard" form which is determined exclusively by a complete system of invariants for the involved equivalence relation. Thus, once these invariants are known, the canonical form is also known.

Canonical representations are very useful in the analysis and design of systems because they display in a conventient format the structure, dynamics, and other important information about the system which they represent.

(2.54) Theorem. Consider an infinite sequence $\{L_1, L_2, \ldots\}$ of $m\times r$ matrices which admits a finite-dimensional realization (2.1), together with its corresponding parameters $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ as in (2.32). Then, the representation of (2.1) given by

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & & A_{rr} \end{bmatrix} \tag{2.55a}$$

(2.55b)
$$A_{ii} = \left[\begin{array}{c|c} \underline{0}^T_{\mu_i-1} & -\gamma_{ii1} \\ \hline & -\gamma_{ii2} \\ I_{\mu_i-1} & \vdots \\ & -\gamma_{ii\mu_i} \end{array}\right]$$

(2.55c)
$$A_{ij} = \left[\begin{array}{c|c} & -\gamma_{ji1} \\ & \vdots \\ 0_{\mu_i,\mu_j-1} & -\gamma_{ji\sigma_{ji}} \\ & 0 \\ & \vdots \\ & 0 \end{array}\right]$$

(2.56a)
$$Q = [\underline{q}_1 \quad \underline{q}_2 \quad \cdots \quad \underline{q}_r]$$

(2.56b)
$$\underline{q}_1 = \underline{i}_1, \ \underline{q}_2 = \underline{i}_{\mu_1+1}, \ \ldots, \ \underline{q}_r = \underline{i}_{\mu_1+\ldots+\mu_{r-1}+1}$$

(2.57a)
$$D = [\underline{d}_{11} \cdots \underline{d}_{1\mu_i} \quad \underline{d}_{21} \cdots \underline{d}_{2\mu_2} \quad \cdots \quad \underline{d}_{r1} \cdots \underline{d}_{r\mu_r}]$$

(2.57b)
$$\underline{d}_{jk} = \underline{\ell}_{jk} \qquad j = 1,2,\ldots r \qquad k = 1,2,\ldots,\mu_j$$

is a canonical state-space representation.

Proof. Notice that the size of all the submatrices of A and the position of unity elements in the A_{ii} submatrices are determined exclusively by the values of the parameters $\{\mu_i\}$. The A_{ii} submatrices are in the well-known observable canonical form, and the A_{ij} submatrices have nonzero elements only in the last column; further, these last-column elements are either given by the negative of one of the $\{\gamma_{ijk}\}$ scalars or zero. Thus, A is completely defined by the $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ invariants. Simi-

larly, Q is determined completely by the $\{\mu_i\}$, since its only nonzero elements have unity value, and their positions in the columns of Q depend exclusively on the values of the $\{\mu_i\}$. The columns of D are grouped in "chains", the length of which is determined by the size of the $\{\mu_i\}$, and their value is given by the $\{\underline{\ell}_{jk}\}$ vectors. Thus, the triple $(A,Q,D)_p$ in (2.55)-(2.57) is determined completely by the $\{\mu_i\}$, $\{\gamma_{ijk}\}$, and $\{\underline{\ell}_{jk}\}$ invariants. Uniqueness of the form is obvious.

That $(A,Q,D)_p$ as above generates the given sequence follows by computing the products (2.4a) and applying (2.32). ∇

A system (2.1) which is represented by (2.55)-(2.57) "looks" like a cascade of single input subsystems coupled in both directions by the last state of each subsystem. The size of subsystem i is μ_i, and its dynamics are given in the roots of its characteristic polynomial:

$$\gamma_{ii}(s) = s^{\mu_i} + \gamma_{ii\mu_i} s^{\mu_i - 1} + \ldots + \gamma_{ii2}s + \gamma_{ii1}$$

It should be noticed that unless the coupling of the subsystems is zero (at least in one direction), the overall system dynamics are not given (in general) by the roots of the characteristic polynomials of the r subsystems. However, since the nonzero elements of A are the $\{\gamma_{ijk}\}$ invariants, they determine the overall system dynamics exclusively. Thus, it can be stated that the $\{\mu_i\}$ define the basic structure of (2.1), and the $\{\gamma_{ijk}\}$ determine its dynamics.

The $\{\underline{\ell}_{jk}\}$ vector invariants represent the gains through which the states appear at the outputs.

(2.58) Remark. Certain relations exist between the invariants of Theorem (2.49). As discussed in Remark (2.50), the number of $\{\gamma_{ijk}\}$ invariants is determined by the values of the $\{\mu_i\}$ invariants, although

their actual values do not depend at all on the $\{\mu_i\}$ invariants. However, the $\{\gamma_{ijk}\}$ and $\{\underline{\ell}_{jk}\}$ invariants are related to each other in the sense that the observability matrix (2.18) with A and D as in (2.55) and (2.57), respectively, must have full rank for some finite δ.

The above ideas are implicitly expressed in (2.22) and (2.32). The integers $\{\sigma_{ij}\}$ (which are defined by the $\{\mu_i\}$ invariants through (2.23)) and the scalars $\{\gamma_{ijk}\}$ determine the length and the coefficients, respectively, of these two sets of recurrence relations, while the vectors $\{\underline{\ell}_{jk}\}$ are the "initial conditions" for these recurrence relations. In other words, given the invariants of Theorem (2.49), the infinite impulse response sequence of the system can be generated. ∇

The state space representation given in Theorem (2.54) has an application in the problem of identifying the parameters of a system for both the cases of deterministic output data (discussed subsequently in this chapter) or noisy output data (Bonivento et al., 1973; Akaike, 1974). The result of Theorem (2.54) can also be advantageously applied to the design of minimal order observers to estimate linear functions of the state of a system (this application is discussed in Chapter 4).

Several alternate state-space canonical forms can be defined for $\mathcal{S}(p)$ under the action of GL(p). Every canonical form has its own characteristics and is useful in certain applications. An extensive discussion of this topic and a list of the most relevant references appear in a recent survey paper by Denham (1974). For the purposes of this work, the form given in Theorem (2.54) seems to be the best suited.

There is an alternate representation of system (2.1) which is also of some interest in this dissertation. It is a factorization of $\Phi(s)$,

the transfer function matrix of system (2.1). This factorization has the form

$$\Phi(s) = \Psi(s)\Gamma^{-1}(s) \tag{2.59}$$

where the polynomial matrices $\Psi(s)$ and $\Gamma(s)$ are $m\times r$ and $r\times r$, respectively. Such a factorization of $\Phi(s)$ is known as a matrix-fraction description of system (2.1), and has been used extensively in the analysis and design of systems (Popov, 1969; Rosenbrock, 1970; Wolovich, 1972a, b, 1973a, b; Wang and Davison, 1973; Dickinson et al., 1974a, b; Rissanen, 1974).

As with state-space descriptions, matrix-fraction descriptions are not unique. It is possible to define an equivalence relation on the set of all polynomial matrix pairs $(\Psi(s),\Gamma(s))$ which satisfy certain conditions, obtain a complete system of invariants, and define a corresponding set of canonical forms. Popov (1969), Rosenbrock (1970), and Dickinson et al. (1974b), among others, have investigated this problem extensively, and it is not considered here. For the purposes of this dissertation it suffices (and seems to be preferable) only to be able to relate in some simple way the canonical state-space description (2.55)-(2.57) of system (2.1) with a matrix-fraction description (not necessarily canonical) of (2.1). However, it is required that $(\Psi(s),\Gamma(s))$ be an irreducible pair (that $\Psi(s)$ and $\Gamma(s)$ have no non-trivial polynomial factors in common). This is the same thing as a state-space description being completely controllable and observable. The matrix-fraction description given in the following theorem satisfies these requirements.

(2.60) Theorem. Consider an infinite sequence $\{L_1, L_2, \ldots\}$ of $m\times r$ matrices which admits a finite-dimensional realization (2.1), together with its corresponding parameters $\{\mu_i\}$

and $\{\gamma_{ijk}\}$ as in (2.32). Then, there is a matrix-fraction description of (2.1) with $\Psi(s)$ and $\Gamma(s)$ given as

(2.61) $$\Gamma(s) = \Gamma_{\mu+1}s^{\mu} + \Gamma_{\mu}s^{\mu-1} + \ldots + \Gamma_2 s + \Gamma_1$$

(2.62a) $$\Gamma(s) = \begin{bmatrix} \gamma_{11}(s) & \gamma_{12}(s) & \ldots & \gamma_{1r}(s) \\ \gamma_{21}(s) & \gamma_{22}(s) & \ldots & \gamma_{2r}(s) \\ \vdots & \vdots & & \vdots \\ \gamma_{r1}(s) & \gamma_{r2}(s) & \ldots & \gamma_{rr}(s) \end{bmatrix}$$

(2.62b) $$\gamma_{ii}(s) = s^{\mu_i} + \gamma_{ii\mu_i}s^{\mu_i-1} + \ldots + \gamma_{ii2}s + \gamma_{ii1}$$

(2.62c) $$\gamma_{ij}(s) = \gamma_{ji\sigma_{ji}}s^{\sigma_{ji}-1} + \gamma_{ji\alpha_{ji}-1}s^{\sigma_{ji}-2} + \ldots$$
$$+ \gamma_{ji2}s + \gamma_{ji1}$$

(2.63) $$\Psi(s) = \Psi_{\mu}s^{\mu-1} + \Psi_{\xi-1}s^{\mu-2} + \ldots + \Psi_2 s + \Psi_1$$

(2.64) $$\Psi_i = \sum_{j=1}^{\mu+1-i} L_j\Gamma_{i+j} \qquad i = 1,2,\ldots\mu$$

This matrix-fraction description is irreducible.

The proof of the above theorem is omitted. Equations (2.61) and (2.64) are easily verified by writing (2.59) as $\Phi(s)\,\Gamma(s) = \Psi(s)$, substituting (2.61) and (2.63) for $\Gamma(s)$ and $\Psi(s)$, equating powers of s on both sides, and finally using the recursive relations (2.32). But to prove that $\Gamma(s)$ is nonsingular and that the pair $(\Psi(s),\Gamma(s))$ is irreducible is more complicated and quite lengthy.

Rosenbrock (1970) gives a detailed proof from a computational point of view starting with his system matrix representation in the state-space canonical form (2.55)-(2.57). In Rosenbrock's work the

above results appear in a slightly different form (he further reduced $\Gamma(s)$ to lower-triangular form by performing column operations on it; of course, the same column operations must be performed on $\Psi(s)$ also).

If the controllability indices are ordered in non-increasing fashion (that is, $\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_r$), then the matrix-fraction description of Theorem (2.60) is a canonical matrix-fraction description (Popov, 1969; Dickinson et al., 1974b). Ordering the controllability indices in such a way corresponds to ordering the inputs. However, such an operation is not an admissible operation for equivalence in $S(p)$ under the action of GL(p), so the meaning of $\{\gamma_{ijk}\}$ and $\{\underline{\ell}_{jk}\}$ as invariants is destroyed. A new system of invariants would have to be defined. This is due to the fact that the group which determines the equivalence relation commonly defined on the set of polynomial matrix pairs $(\Psi(s), \Gamma(s))$ is not GL(p) (see Popov, 1969; Rosenbrock, 1970; or Dickinson et al., 1974a,b).

Rissanen (1974) forces $\Gamma_{\mu+1}$ to be nonsingular and obtains a canonical form closely related to the form described in Theorem (2.60). However, as pointed out by Dickinson et al. (1974a,b), this condition is unnecessary and (generally) gives a reducible form for the pair $(\Psi(s),\Gamma(s))$.

The canonical form of Theorem (2.54) is related in a simpler fashion to the system of invariants $\{\mu_i\}$, $\{\gamma_{ijk}\}$, and $\{\underline{\ell}_{jk}\}$ than the matrix-fraction description of Theorem (2.60). Thus, it seems reasonable that (2.55)-(2.57) would be a more useful representation of (2.1) than (2.61)-(2.64) for most applications. This indeed turns out to be the case in the minimal complete and partial realization problems and in the problems discussed in the following chapters. Mostly due to this, the results that follow are developed in the context of state-space

descriptions only. However, since the results are obtained in terms of the invariants $\{\mu_i\}$, $\{\gamma_{ijk}\}$, and $\{\underline{\ell}_{jk}\}$, then, in view of (2.61)-(2.64), a matrix-fraction description is also practically available.

Minimal Realization Algorithms

Lemma (2.21), Corollary (2.31), and Theorems (2.49), (2.54) provide the basis for an attractive solution to the minimal complete and the minimal partial realization problems. Such a solution is presented after the following discussion.

Recall from Definition (2.4) that in the minimal complete realization problem one is faced with the prospect of examining an infinite amount of data. Thus, strictly speaking, the minimal complete realization problem can never actually be solved. Fortunately, the problem can be proposed in a more ameanable form with the help of the following definition.

(2.65) Definition. Consider an infinite sequence $\{L_1, L_2, ...\}$ of $m\times r$ matrices. A matrix triple $(A,Q,D)_p$ is said to be a minimal realization of degree N_0 of the given sequence if and only if $(A,Q,D)_p$ is a minimal partial realization of $\{L_1, L_2 ... L_{N_0}\}$.†

Now the minimal realization problem can be considered to be that of obtaining a triple $(A,Q,D)_p$ which satisfies conditions (2.5a) and (2.5b) as N_0 is increased. The best way to solve such a problem seems to be one which is recursive. Here recursive means that as N_0 is

†Standard usage (Kalman et al., 1969; Kalman, 1971) dictates that a realization which satisfies (2.5a) and (2.5b) is one of order N_0. The above change in nomenclature is introduced here to avoid confusion in later chapters where (in the context of observer design) order of a minimal realization refers to its dimension.

increased to N_0', the computations used to obtain the triple $(A,Q,D)_{p_M}$ which realizes the given sequence up to the N_0th element are part of the computations required to obtain the triple $(A,Q,D)_{p_M'}$ which realizes the given sequence up to the N_0'th element. Then only a few numbers need to be calculated every time that a new data point (an L_i matrix) forces the dimension of the minimal realization to increase.

Rissanen (1971) has given one such recursive solution in the state-space representation. His algorithm consists of a special factorization of the Hankel array $\mathcal{H}_{NN'}$ into a lower-triangular matrix $\tilde{L}$ that has 1's in the main diagonal, and another matrix $\tilde{U}$ which is in a special form. This factorization allows matrices Q and D to be obtained by inspection, and matrix A is obtained by inverting a lower-triangular submatrix of $\tilde{L}$. As new data is added, the size of $\mathcal{H}_{NN'}$ increases, and so does the size of $\tilde{L}$ and $\tilde{U}$. However, the top portion of matrices $\tilde{L}$ and $\tilde{U}$ is available from previous computations, so only the bottom elements have to be calculated. Everytime the new data increases the rank of $\mathcal{H}_{NN'}$, a new realization has to be obtained. But due to the factorization procedure used, the matrices of the previous realizations are upper-left-hand corner submatrices of the new realization. Thus, Rissanen's algorithm is a truly recursive procedure.

The algorithm given below is not totally recursive, but it is shown that there is some recursiveness involved. The procedure followed is to first identify the invariants $\{\mu_i\}$, $\{\gamma_{ijk}\}$, and $\{\underline{\ell}_{jk}\}$ of the realization of degree N_0. Then N_0 is increased to N_0' and the invariants of the realization of degree N_0' are identified. Some of the invariants do not change as the minimal realization increases in size due to the larger number of impulse sequence terms being considered. This is the

partial recursive property previously mentioned. Further, since the invariants of the realization are identified, the algorithm also has the convenient feature that the successive realizations are obtained in the canonical form (2.55)-(2.57).

(2.66) Minimal Complete Realization Algorithm.

Step 1. Let N and N' be the smallest integers for which the realizability criterion (2.10) is satisfied with i = j = 1. That is,

(2.67) $$\rho(\mathcal{H}_{NN'}) = \rho(\mathcal{H}_{NN'+1}) = \rho(\mathcal{H}_{N+1N'}) = p_M$$

Also, define $N_0 = N + N'$.

Step 2. Consider now the Hankel array $\mathcal{H}_{NN'+1}$. Following Definition (2.8), denote its (r(k-1)+j)th column by $\underline{h}_{jk}$. For notational simplicity, assume that $\underline{h}_{11}$ is not a null vector. Starting with $\underline{h}_{21}$, examine (in order) the columns of $\mathcal{H}_{NN'+1}$ for linear dependence upon the preceding columns, and keep the linearly independent ones (including $\underline{h}_{11}$). Take the selected columns and rearrange their order so that the second subscript varies the fastest. This result in the following ordering

(2.68) $$\underline{h}_{11}, \underline{h}_{12}, \ldots, \underline{h}_{1\mu_1}, \underline{h}_{21}, \ldots, \underline{h}_{2\mu_2}, \ldots, \underline{h}_{r1}, \ldots, \underline{h}_{r\mu_r}$$

The integers $\{\mu_i\}$ which are implicitly defined in (2.68) are the controllability indices of the minimal realization of degree N_0. Let L be a full rank $mN \times p_M$ matrix whose columns are the vectors in (2.68) in that order. Notice that $p_M \leqq mN$, or else the selected columns are not linearly independent. Notice also that L has N block rows of p_M block elements each, and these block elements are columns of the matrices in the given sequence.

Step 3. The vectors $\underline{h}_{1\mu_1+1}, \underline{h}_{2\mu_2+1}, \ldots, \underline{h}_{r\mu_r+1}$ are linearly dependent on the vectors in (2.68). More precisely, due to the selection procedure used, $\underline{h}_{i\mu_i+1}$ for i = 1,2,...,r, is linearly dependent only on the following columns of L

$$\begin{array}{ll} \underline{h}_{i1}, \underline{h}_{i2}, \ldots, \underline{h}_{i\mu_i} & \\ \underline{h}_{j1}, \underline{h}_{j2}, \ldots, \underline{h}_{j\mu_i} & \text{for } \mu_i \leqq \mu_j \text{ and } i < j \\ \underline{h}_{j1}, \underline{h}_{j2}, \ldots \underline{h}_{j\mu_i+1} & \text{for } \mu_i < \mu_j \text{ and } i > j \\ \underline{h}_{j1}, \underline{h}_{j2}, \ldots, \underline{h}_{j\mu_j} & \text{for } \mu_i \geqq \mu_j \text{ and } i \neq j \end{array} \tag{2.69}$$

where j = 1,2,...,r also. Notice that the length of the vector "chains" in (2.69) is determined by the second subscript on the vectors, and this subscript is equal to σ_{ij} calculated according to (2.23). The dependency of $\underline{h}_{i\mu_i+1}$ on these columns of L can be written compactly as

$$\underline{h}_{i\mu_i+1} = -\hat{L}\underline{\gamma}_i \qquad i = 1,2,\ldots,r \tag{2.70}$$

where the hat (^) over matrix L denotes that some columns of L are not (generally) included, and $\underline{\gamma}_i$ is the following σ_i-dimensional vector (σ_i is as defined in (2.51), and $\sigma_i \leqq p_M$)

$$\underline{\gamma}_i^T = [\gamma_{i11} \quad \gamma_{i12} \quad \ldots \quad \gamma_{ir\sigma_{ir}}] \qquad i = 1,2,\ldots,r \tag{2.71}$$

Equation (2.70) is now solved for the unknown $\underline{\gamma}_i$. This gives

$$\underline{\gamma}_i = - \hat{L}^{\#}\underline{h}_{i\mu_i+1} \qquad i = 1,2,\ldots,r \tag{2.72}$$

where $\hat{L}^{\#} = (\hat{L}^T\hat{L})^{-1}\hat{L}^T$ is the (left) pseudoinverse of $\hat{L}$.

Step 4. A minimal realization of degree No of the given sequence is now constructed in the form (2.55)-(2.57) using indices $\{\mu_i\}$ identified in Step 2, the coefficients $\{\gamma_{ijk}\}$ identified in Step 3, and the p_M m-dimensional vectors $\{\underline{\ell}_{jk}\}$ which make up the first block row of L.

Step 5. Add new data to the truncated Hankel array $\mathcal{H}_{NN'}$ until the rank is found to increase and the realizability criterion is met again for some new integers N and N', and $N_0' = N + N'$. Repeat Steps 2-4 to obtain a new realization of degree N_0'. And so on. This cycle can conceivably continue infinitely many times, because the given sequence is infinite. One method of checking when to stop is to compute a realization of some degree, say N_0', and use (2.32) to compare the terms of the impulse response sequence which are after the N_0'th one with the corresponding terms in the sequence being realized.

Proof. Kalman has shown (see Kalman et al., 1969) that (2.67) is satisfied if and only if there exists a triple $(A,Q,D)_{p_M}$ such that (2.5a) is true. Thus, it suffices to show that a realization constructed using the algorithm has dimension p_M and satisfies (2.5a). This is done by showing that the indices $\{\mu_i\}$, the coefficients $\{\gamma_{ijk}\}$, and the vectors $\{\underline{\ell}_{jk}\}$ identified in the algorithm are the invariants of the minimal realization of the matrix sequence $\{L_1, L_2, \ldots, L_{N_0}\}$.

The procedure used to select the basis (2.68) defines the integers $\{\mu_i\}$ uniquely and precisely. The number of vectors in (2.68) is p_M. Thus, it follows that $p_M = \mu_1 + \mu_2 + \ldots + \mu_r$; further, $N' = \max(\mu_i)$. Also, in view of (2.4a) and the definition of $\underline{h}_{jk}$,

$$(2.73) \qquad \underline{h}_{jk} = \begin{bmatrix} \underline{\ell}_{jk} \\ \underline{\ell}_{jk+1} \\ \cdot \\ \cdot \\ \cdot \\ \underline{\ell}_{jk+N-1} \end{bmatrix} = \begin{bmatrix} DA^{k-1}\underline{q}_j \\ DA^{k}\underline{q}_j \\ \cdot \\ \cdot \\ \cdot \\ DA^{k+N-2}\underline{q}_j \end{bmatrix} \qquad \begin{matrix} j = 1,2,\ldots,r \\ k = 1,2,\ldots,\mu_j \end{matrix}$$

and, correspondingly,

$$(2.74)\qquad \underline{h}_{i\mu_i+1} = \begin{bmatrix} \underline{\ell}_{i\mu_i+1} \\ \underline{\ell}_{i\mu_i+2} \\ \cdot \\ \cdot \\ \cdot \\ \underline{\ell}_{i\mu_i+N} \end{bmatrix} = \begin{bmatrix} DA^{\mu_i}\underline{q}_i \\ DA^{\mu_i+1}\underline{q}_i \\ \cdot \\ \cdot \\ \cdot \\ DA^{\mu_i+N-1}\underline{q}_i \end{bmatrix} \qquad i = 1,2,\ldots,r$$

The above discussion, equation (2.70), and Corollary (2.31) imply that the $\{\mu_i\}$ are invariants of the minimal realization of $\{L_1,L_2,\ldots,L_{N_0}\}$, and that N' is the controllability index of the realization.

Matrix $\hat{L}$ in (2.70) has full rank and $\underline{h}_{i\mu_i+1}$ is in its column space (by construction); thus, $\underline{\gamma}_i$ as given in (2.72) is unique. It follows from this and Corollary (2.31) that the vectors $\{\underline{\gamma}_i\}$ are also invariants of the minimal realization of the sequence $\{L_1,L_2,\ldots,L_{N_0}\}$.

Finally, the vectors $\{\underline{\ell}_{jk}\}$ that comprise the first block row of L are also invariants of the minimal realization of $\{L_1,L_2,\ldots,L_{N_0}\}$, since they are part of the first p_M linearly independent columns of $\mathcal{H}_{NN'+1}$.

Once the realization invariants are known, the canonical form (2.55)-(2.57) is easily constructed. That (2.5a) is satisfied follows from (2.32) and Theorem (2.54). ∇

(2.75) Remark. In the context of the minimal realization problem, consider the case where a minimal realization of degree N_0 has been obtained, and suppose that as more data is added to the Hankel array, the rank of the array increases from the value p_M. Suppose further that the realizability criterion has been met anew for the data up to the N_0'th element. The effect of the $N_0' - N_0$ added data points is to add to the basis (2.68) some vectors which were not included previously, and

to add more elements at the bottom of those vectors which were already in the basis. As a result, the number of columns in the first block row of L increases. In conclusion, it is observed that if the new data points cause the dimension of the minimal realization to increase, then the set of invariant vectors $\{\underline{\ell}_{jk}\}$ is larger, with the further condition that the vectors which were previously in the set remain in the set. If a prime(') denotes the invariants for the realization of degree N_0' and the unprimed quantities correspond to the previous realization of degree N_0, then the above result can be stated (using set notation) as follows: $\{\underline{\ell}_{jk}\} \subset \{\underline{\ell}'_{jk}\}$.

Unfortunately, it is not true that $\{\mu_i\} \subset \{\mu_i'\}$ and $\{\gamma_{ijk}\} \subset \{\gamma'_{ijk}\}$. It follows from Definition (2.19) that as p_M increases, at least one (possibly all) of the $\{\mu_i\}$ indices has to increase. By the same token, the size of L increases and the vector $\underline{h}_{i\mu_i+1}$ in the left-hand side of (2.70) changes for at least one (possibly all) value of i. Thus, at every iteration (Step 5) it may be necessary to compute again one or more of the vectors $\{\underline{\gamma}_i\}$. However, it is true that in some cases subsets of the sets $\{\mu_i\}$ and $\{\underline{\gamma}_i\}$ remain the same from one iteration to the next. For these reasons the realization algorithm proposed above can be labeled as quasi-recursive. The example given below illustrates the application of the algorithm and the quasi-recursive property. ∇

Bonivento et al. (1973), Rissanen (1974), and Akaike (1974) have (independently) proposed realization algorithms similar to the one given above, but they do not discuss the quasi-recursive property and some of the other significant points mentioned above. Dickinson et al. (1974a,b) discuss a related quasi-recursive algorithm from the matrix-fraction representation point of view. The procedure they suggest is to modify

the realization for every data point that does not "fit" into the current pattern, rather than waiting to have enough data points so that the realizability criterion is met anew, which is the procedure suggested in Algorithm (2.66). Their approach seems to be more desirable if the data is available sequentially, while the one given here seems to be more convenient if the complete sequence is available at a time. The algorithm presented above (as well as most of the results which preceded it and follow next) was derived independently of the results in the above-mentioned references.

(2.76) Example. It is desired to obtain the minimal realization of an infinite sequence which begins with the following elements:

$$L_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad L_2 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \quad L_3 = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix}$$

$$L_4 = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \quad L_5 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \quad L_6 = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}$$

$$L_7 = \begin{bmatrix} 2 & 6 \\ 0 & 7 \end{bmatrix} \quad L_8 = \begin{bmatrix} 0 & -14 \\ -1 & -5 \end{bmatrix} \quad L_9 = \begin{bmatrix} -1 & 29 \\ -1 & 27 \end{bmatrix}$$

$$L_{10} = \begin{bmatrix} -2 & -80 \\ 0 & -50 \end{bmatrix} \quad L_{11} = \begin{bmatrix} 0 & 157 \\ 1 & 110 \end{bmatrix} \quad L_{12} = \begin{bmatrix} 1 & -381 \\ 1 & -281 \end{bmatrix}$$

$$L_{13} = \begin{bmatrix} 2 & 879 \\ 0 & 580 \end{bmatrix} \quad L_{14} = \begin{bmatrix} 0 & -1928 \\ -1 & -1397 \end{bmatrix}$$

The partial Hankel array $\mathcal{H}_{55}$ for the above sequence is

$$\mathcal{H}_{55} = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 2 & -2 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 & -2 & 1 & -1 \\ 0 & 1 & -1 & 1 & -2 & 1 & 0 & 1 & 1 & -3 \\ -1 & 1 & -1 & 0 & 0 & -2 & 1 & -1 & 1 & -2 \\ -1 & 2 & -2 & 1 & 0 & 1 & 1 & -3 & 2 & 6 \\ -1 & 0 & 0 & -2 & 1 & -1 & 1 & -2 & 0 & 7 \\ -2 & 1 & 0 & 1 & 1 & -3 & 2 & 6 & 0 & -14 \\ 0 & -2 & 1 & -1 & 1 & -2 & 0 & 7 & -1 & -5 \\ 0 & 1 & 1 & -3 & 2 & 6 & 0 & -14 & -1 & 29 \\ 1 & -1 & 1 & -2 & 0 & 7 & -1 & -5 & -1 & 27 \end{bmatrix}$$

It is found that the realizability criterion is met first with $N = N' = 1$, and $p_M = 2$. Inspection of $\mathcal{H}_{12}$ gives $\mu_1 = \mu_2 = 1$ and

$$L = [\underline{h}_{11} \quad \underline{h}_{21}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\underline{\ell}_{11} \quad \underline{\ell}_{21}]$$

The coefficient invariants are obtained next as

$$\underline{\gamma}_1 = \begin{bmatrix} \gamma_{111} \\ \gamma_{121} \end{bmatrix} = -L^{-1}\underline{h}_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{\gamma}_2 = \begin{bmatrix} \gamma_{211} \\ \gamma_{221} \end{bmatrix} = -L^{-1}\underline{h}_{22} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

The state-space canonical representation of the minimal realization of degree $N_0 = N + N' = 2$ is now easily obtained as

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the matrix-fraction representation obtained using (2.61)-(2.64) is given by

$$\Phi(s) = \Psi(s)\Gamma^{-1}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ 1 & s-1 \end{bmatrix}^{-1}$$

The poles of the realization are at $0.5 \pm j0.866$.

The realizability criterion is met next for $N = N' = 2$, and $p_M = 3$. The linearly independent columns of $\mathcal{H}_{23}$ are $\underline{h}_{11}$, $\underline{h}_{21}$, and $\underline{h}_{22}$, so $\mu_1 = 1$, $\mu_2 = 2$, and

$$L = [\underline{h}_{11} \quad \underline{h}_{21} \quad \underline{h}_{22}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\ell}_{11} & \underline{\ell}_{21} & \underline{\ell}_{22} \\ \underline{\ell}_{12} & \underline{\ell}_{22} & \underline{\ell}_{23} \end{bmatrix}$$

Column $\underline{h}_{12}$ can depend only on $\underline{h}_{11}$ and $\underline{h}_{21}$, so

$$\underline{\gamma}_1 = -\hat{L}^{\#}\underline{h}_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma_{111} \\ \gamma_{121} \end{bmatrix}$$

Column $\underline{h}_{23}$ can depend on all three vectors in L, so

$$\underline{\gamma}_2 = -L^{\#}\underline{h}_{23} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \gamma_{211} \\ \gamma_{221} \\ \gamma_{222} \end{bmatrix}$$

Then the state-space and matrix-fraction representations of the minimal realization of degree $N_0 = N + N' = 4$ are obtained as

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Phi(s) = \Psi(s)\Gamma^{-1}(s) = \begin{bmatrix} 1 & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} s & -1 \\ 1 & s^2-s+1 \end{bmatrix}^{-1}$$

and the poles of the realization are (approximately) at: $0.772 \pm j1.12$, -0.544.

It is important to observe that several realization invariants remain the same in value in passing from the realization of degree 2 to that of degree 4. These are: μ_1, γ_{111}, γ_{121}, $\underline{\ell}_{11}$, and $\underline{\ell}_{12}$. Since μ_1 and $\underline{\gamma}_1$ do not change, the calculations to obtain $\underline{\gamma}_1$ are unnecessary.

Examining the columns of $\mathcal{H}_{55}$, $\mathcal{H}_{56}$, and $\mathcal{H}_{65}$ for linear dependency, it is found that the realizability criterion is satisfied next for N = N' = 5, and $p_M = 7$. The linearly independent columns of $\mathcal{H}_{56}$ are found to be $\underline{h}_{11}$, $\underline{h}_{21}$, $\underline{h}_{12}$, $\underline{h}_{22}$, $\underline{h}_{13}$, $\underline{h}_{23}$, and $\underline{h}_{14}$. Therefore, $\mu_1 = 4$, $\mu_2 = 3$, and

$$L = [\underline{h}_{11} \quad \underline{h}_{12} \quad \underline{h}_{13} \quad \underline{h}_{14} \quad \underline{h}_{21} \quad \underline{h}_{22} \quad \underline{h}_{23}]$$

$$= \begin{bmatrix} \underline{\ell}_{11} & \underline{\ell}_{12} & \underline{\ell}_{13} & \underline{\ell}_{14} & \underline{\ell}_{21} & \underline{\ell}_{22} & \underline{\ell}_{23} \\ \underline{\ell}_{12} & \underline{\ell}_{13} & \underline{\ell}_{14} & \underline{\ell}_{15} & \underline{\ell}_{22} & \underline{\ell}_{23} & \underline{\ell}_{24} \\ \underline{\ell}_{13} & \underline{\ell}_{14} & \underline{\ell}_{15} & \underline{\ell}_{16} & \underline{\ell}_{23} & \underline{\ell}_{24} & \underline{\ell}_{25} \\ \underline{\ell}_{14} & \underline{\ell}_{15} & \underline{\ell}_{16} & \underline{\ell}_{17} & \underline{\ell}_{24} & \underline{\ell}_{25} & \underline{\ell}_{26} \\ \underline{\ell}_{15} & \underline{\ell}_{16} & \underline{\ell}_{17} & \underline{\ell}_{18} & \underline{\ell}_{25} & \underline{\ell}_{26} & \underline{\ell}_{27} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & -1 & -2 & 0 & 1 & 2 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 & 1 & 2 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 & -2 \\ -1 & -2 & 0 & 1 & 2 & 1 & 1 \\ -1 & 0 & 1 & 1 & 0 & -2 & -1 \\ -2 & 0 & 1 & 2 & 1 & 1 & -3 \\ 0 & 1 & 1 & 0 & -2 & -1 & -2 \\ 0 & 1 & 2 & 0 & 1 & -3 & 6 \\ 1 & 1 & 0 & -1 & -1 & -2 & 7 \end{bmatrix}$$

Both $\underline{h}_{15}$ and $\underline{h}_{24}$ can depend on all the vectors in L, so $\underline{\gamma}_1 = -L^{\#}\underline{h}_{15}$ and $\underline{\gamma}_2 = -L^{\#}\underline{h}_{24}$. The values of $\underline{\gamma}_1$ and $\underline{\gamma}_2$ are found to be

$$\underline{\gamma}_1^T = [\gamma_{111} \quad \gamma_{112} \quad \gamma_{113} \quad \gamma_{114} \quad \gamma_{121} \quad \gamma_{122} \quad \gamma_{123}]$$

$$= [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$\underline{\gamma}_2^T = [\gamma_{211} \quad \gamma_{212} \quad \gamma_{213} \quad \gamma_{214} \quad \gamma_{221} \quad \gamma_{222} \quad \gamma_{223}]$$

$$= [6 \quad 0 \quad 0 \quad 2 \quad 5 \quad -3 \quad 0]$$

Now the canonical state-space and matrix-fraction representations of the minimal realization of degree $N_0 = N + N' = 10$ are easily constructed. They are given by

$$A = [\underline{i}_2 \quad \underline{i}_3 \quad \underline{i}_4 \quad -\underline{\gamma}_1 \quad \underline{i}_6 \quad \underline{i}_7 \quad -\underline{\gamma}_2] \qquad Q = [\underline{i}_1 \quad \underline{i}_5]$$

$$D = [\underline{\ell}_{11} \quad \underline{\ell}_{12} \quad \underline{\ell}_{13} \quad \underline{\ell}_{14} \quad \underline{\ell}_{21} \quad \underline{\ell}_{22} \quad \underline{\ell}_{23}]$$

$$\Phi(s) = \Psi(s)\Gamma^{-1}(s) = \begin{bmatrix} s^3-s-1 & 2s^2+s \\ -s^2-s & s^2-s-5 \end{bmatrix} \begin{bmatrix} s^4+s & 2s^3+6 \\ 0 & s^3-3s+5 \end{bmatrix}^{-1}$$

The poles of the realization are (approximately) at: 0.0, -1.0, -2.28, $0.5 \pm j0.866$, $1.14 \pm j0.946$.

As expected, the only invariants that do not change in value in passing from the preceding realization are: $\underline{\ell}_{11}$, $\underline{\ell}_{21}$, and $\underline{\ell}_{22}$.

Using (2.32) it is found that the elements L_{11}, L_{12}, L_{13}, and L_{14} of the impulse response sequence of the above realization agree with the corresponding elements of the given matrix sequence. Thus, the above realization is the minimal realization (of degree 14) of the given sequence. ∇

(2.77) Remark. In the process of executing Steps 1 and 5 in Algorithm (2.66), if an appropriate numerical procedure is used, it is possible to obtain the invariants $\{\gamma_{ijk}\}$ concurrently with the invariants $\{\mu_i\}$ and $\{\ell_{jk}\}$ when checking the columns of the truncated Hankel array for

linear dependencies. This problem is currently being investigated by Candy (1975), and preliminary results show an interesting relationship with the previous work of Rissanen (1971).

An algorithm to solve the minimal partial realization problem is presented next. The algorithm has the same convenient features of Algorithm (2.66); that is, the realization invariants are identified and the realization is obtained in canonical form. Further, this algorithm can be used as an alternate constructive proof for Theorem (2.16), which has some advantages over the proof given by Kalman (1971b) and Tether (1970). This algorithm is also the subject of a forthcoming paper by Roman and Bullock (1975).

(2.78) Minimal Partial Realization Algorithm.

Step 1. Form the truncated generalized Hankel matrix $\mathcal{H}_{N_0 N_0}$ (the $mN_0 \times rN_0$ upper left-hand block of $\mathcal{H}$ in (2.11)). As before, let $\underline{h}_{jk}$ be the $(r(k-1)+j)$th column of $\mathcal{H}_{N_0 N_0}$, and suppose that $\underline{h}_{11}$ is not a null vector (for notational simplicity). Starting with $\underline{h}_{21}$, examine (in order) the fixed portion (specified elements) of the columns of $\mathcal{H}_{N_0 N_0}$ for linear dependence upon the corresponding portion of the preceding columns, and keep the linearly independent columns (including $\underline{h}_{11}$). That is, if the fixed portion of $\underline{h}_{jk}$ (which is denoted as $\hat{\underline{h}}_{jk}$) is linearly independent of the corresponding portion of the same length of its preceding columns, then $\underline{h}_{jk}$ is kept; otherwise, it is rejected. Take the selected columns and rearrange their order so that the second subscript varies the fastest. This results in the ordering (2.68). The integers $\{\mu_i\}$ which are implicitly defined in (2.68) are the controllability indices of the minimal

realization of the given partial sequence, and the dimension of the realization is $p_M = \mu_1 + \mu_2 + \mu_r$.

Step 2. The fixed portion of the vectors $\underline{h}_{1\mu_1+1}, \underline{h}_{2\mu_2+1}, \ldots, \underline{h}_{r\mu_r+1}$ are of length $m(N_0-\mu_i)$ for $i = 1,2,\ldots,r$, and are linearly dependent on the corresponding portions of the same length of the vectors in (2.68). More precisely, due to the selection procedure, used, $\underline{h}_{i\mu_i+1}$ for $i = 1,2,\ldots,r$, is linearly dependent on the corresponding portions of the same length of the vectors in (2.69). As before, the length of the "chains" in (2.69) is equal to σ_{ij} calculated according to (2.23). Also, $\sigma_i \leq p_M$ (with σ_i defined as in (2.51)). If L is the $mN_0 \times p_M$ matrix whose columns are the vectors in (2.68) in that order, and M_i is the $m(N_0-\mu_i)\times p_M$ top portion of L, then the dependency of $\hat{\underline{h}}_{i\mu_i+1}$ for $i = 1,2,\ldots,r$, on the corresponding portions of the same length of the vectors in (2.69) can be written as

(2.79) $$\hat{\underline{h}}_{i\mu_i+1} = -\hat{M}_i\underline{\gamma}_i \qquad i = 1,2,\ldots,r$$

where the hat (^) over matrix M_i denotes that some columns of M_i are not (generally) included, and $\underline{\gamma}_i$ is as defined in (2.71). $\hat{M}_i$ is an $m(N_0-\mu_i)\times\sigma_i$ matrix and, depending on the actual values of the given sequence, the value of $m(N_0-\mu_i)$ may be less than, equal to, or greater than the value of σ_i. By construction, a solution vector $\underline{\gamma}_i$ for (2.79) exists, but it is possible that it may not be unique. Nonuniqueness results whenever $\rho(\hat{M}_i) < \sigma_i$ (more unknowns than linearly independent equations), and the number of degrees of freedom available is given by

(2.80) $$\sum_{i=1}^{r} (\sigma_i - \rho(\hat{M}_i)) = \sigma - \sum_{i=1}^{r} \rho(\hat{M}_i)$$

Step 3. The minimal partial realization of the given sequence has dimension p_M, and the triple $(A,Q,D)_{p_M}$ is constructed in the form (2.55)-(2.57) using the indices $\{\mu_i\}$ identified in Step 1, the coefficients $\{\gamma_{ijk}\}$ identified in Step 3 (some of which may be arbitrary), and the p_M m-dimensional vectors $\{\underline{\ell}_{jk}\}$ which make up the first block row of L.

Proof. The proof is based on the discussion which follows equation (2.11) and on the proof of Algorithm (2.66).

Notice that the procedure used to select the linearly independent vectors $\underline{h}_{jk}$ in Step 1 is equivalent to the procedure used to compute p_M according to (2.13). Consequently, the set of integers $\{\mu_i\}$ obtained in Step 1 is the set of controllability indices of any minimal realization of the given sequence. It follows that $p_M = \mu_1 + \mu_2 + \ldots + \mu_r$. Notice also that $\mu = \max(\mu_i) \leq N_0$.

The fixed portion of the vectors $\underline{h}_{i\mu_i+1}$ for $i = 1,2,\ldots,r$, is linearly dependent on the corresponding portions of the same length of the vectors in (2.69); thus, in view of the definition of M_i, (2.79) has at least one solution vector $\underline{\gamma}_i$. The solution is unique if and only if $\rho(M_i) = \sigma_i$, and nonunique if and only if $\rho(\dot{M}_i) < \sigma_i$ (where σ_i is as defined in (2.51), and the integers $\{\sigma_{ij}\}$ are defined by (2.33) in terms of the $\{\mu_i\}$ invariants). Only those scalars γ_{ijk} which are uniquely specified are invariants of the minimal partial realization; those which are arbitrary cannot be called invariants. $\hat{M}_i$ is dimensioned $m(N_0-\mu_i)\times\sigma_i$. It follows that $\hat{M}_i$ is not defined for $\mu_i = N_0$. This corresponds to the case where the left-hand side of (2.79) is unspecified, and therefore, $\rho(\hat{M}_i) = 0$ must be used in (2.80).

Notice also that the vectors $\{\underline{\ell}_{jk}\}$ which comprise the first block row of L are also invariants of the minimal partial realization.

Suppose now that either the solution to (2.79) is unique for all values of i, or that, if nonunique for some values of i, the arbitrary γ_{ijk} have been designated a value according to some criterion. Either way, equation (2.32) is satisfied for values of τ in the range $0 \leq \tau \leq N_0 - \mu_i$. Now substitute the parameters $\{\mu_i\}$, $\{\gamma_{ijk}\}$, and the available vectors $\{\underline{\ell}_{jk}\}$ in (2.32) for $\tau > N_0 - \mu_i$, $i = 1,2,\ldots,r$, to construct the elements of an extension sequence. It follows that such an extension satisfies (2.11) for all values of i and j, and is, therefore, a minimal extension sequence.

Finally, the canonical form (2.55)-(2.57) is easily constructed with the identified invariants. ∇

It should be emphasized that (in both Algorithms (2.66) and (2.78)) it is not mandatory to construct the minimal realization in the canonical form of Theorem (2.54). Any matrix triple which satisfies (2.32) with $DA^{j-1}\underline{q}_k$ in place of $\underline{\ell}_{jk}$ is equally acceptable as a minimal realization. Form (2.55)-(2.57) is adopted here for convenience and simplicity.

Another interesting point is that Algorithm (2.66) can be viewed as a successive application of Algorithm (2.78) in the special case where (2.79) has a unique solution for all values of i. The following remarks are also noteworthy.

(2.81) Remark. The construction of a minimal extension sequence outlined in the proof of Algorithm (2.78) serves as a proof for Theorem (2.13) (due to Kalman, 1971b, and Tether, 1970). Besides being simple to apply, Algorithm (2.78) displays the structure and invariants which are common to all minimal realizations of the given partial sequence, and lumps all the available degrees of freedom in matrix A. These

properties allow the possibility of constructing the minimal realization which has the most convenient set of poles or the most desirable structure (even both in some cases) among all the possible minimal realizations, in a manner much simpler than with the constructions used by Kalman (1971b) and Tether (1970). ∇

(2.82) Remark. As previously mentioned, the state-space canonical form of Theorem (2.54) is not the only canonical form that can be defined for $S(p)$ under the action of GL(p). Other complete systems of invariants can be obtained by examining the columns (rows) of $\mathcal{H}$ for linear dependencies according to a rule different from the one discussed here (see, for example, Denham, 1974). However, it is well known (Kalman, 1971a; Rosenbrock, 1970) that the $\{\mu_i\}$ invariants identified here are the unique set of integers which determines the size of the smallest possible cyclic subsystems into which the system can be divided using state feedback, put a lower bound on the degrees of the invariant factor polynomials that can be attained using state feedback, and define the size of the smallest controllability subspaces that can be attained for a system with state feedback (for the definition and an extensive discussion of controllability subspaces see, for example, Warren and Eckberg, 1974, or the references therein). For these and other related reasons, the controllability indices seem to be the set of integers which best define the basic structure of a system. Analogously, the associated parameters $\{\gamma_{ijk}\}$ and vectors $\{\underline{\ell}_{jk}\}$ contribute to completely define a system in its most basic structure.

The canonical form (2.55)-(2.57) constructed with these invariants has $\sigma + pm \leqq p(r + m)$ non-trivial elements (different from 0's and 1's). This is the smallest number of non-trivial elements needed to represent

the basic structure of a system. From Theorem (2.49) and Remark (2.58) it follows that it is not possible to represent a system with a smaller number of non-trivial elements by using other state-space canonical forms. Also, the structure obtained with another canonical form can be quite different from the actual basic structure of the system. Some of the other canonical forms often lead to non-general and even misleading results when used in the minimal partial realization problem context. This topic is discussed further next. ∇

(2.83) Remark. Ackermann (1972) has proposed an algorithm to solve the minimal partial realization problem using a procedure different from the one discussed here. His procedure consists of examining the columns of $\mathcal{H}_{N_0 N_0}$ (actually, he examines the rows of $\mathcal{H}_{N_0 N_0}$, but the dual of his method is considered here for notational convenience and for comparison purposes) for linear dependency in the following order

(2.84) $$\underline{h}_{11}, \underline{h}_{12}, \ldots, \underline{h}_{1N_0}, \underline{h}_{21}, \underline{h}_{22}, \ldots, \underline{h}_{2N_0}, \ldots, \underline{h}_{r1}, \underline{h}_{r2}, \ldots, \underline{h}_{rN_0}$$

and retaining the columns found to be linearly independent of their preceding columns. Then, following the same lines as in this dissertation, his procedure leads to a canonical form of the type (2.55)-(2.57), but which differs in that the size of the subsystems is generally larger, and the subsystems are coupled only in the backward direction (that is, $A_{ij} = 0$ for $i > j$).

Recently, Ledwich and Fortmann (1974) have shown by examples that the partial realization algorithm of Ackermann (1972) often gives non-minimal and/or misleading results. These cases can occur only if the given partial sequence does not satisfy the realizability criterion, because, otherwise, the minimal partial realization is unique. The

theory discussed in this work and Algorithm (2.78) provide a lucid explanation for the ideas that Ledwich and Fortmann (1974) attempted to convey via examples.

The partial realization algorithm of Ackermann (1972) is based on the minimal complete realization algorithm of Ackermann and Bucy (1971). More recently, Mayne (1972) and Mital and Chen (1973) have also addressed (independently) the minimal realization problem via the same approach. It should be mentioned that in the minimal complete realization problem context, the algorithms lead (in all cases) to realizations which are truly minimal. This is due to the fact that if the given sequence admits a finite-dimensional realization, then the realizability criterion is met and the minimal realization (of dimension equal to the rank of the Hankel matrix) is unique. ∇

The following example (due to Ledwich and Fortmann, 1974) illustrates the application of Algorithm (2.78) and several of the ideas discussed above.

(2.85) Example. It is desired to obtain a minimal realization for the following partial sequence

$$L_1 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad L_2 = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \quad L_3 = \begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix}$$

The partial Hankel array $\mathcal{H}_{33}$ is

$$\mathcal{H}_{33} = \begin{bmatrix} 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 & 0 & 1 \\ 1 & 1 & 3 & 3 & * & * \\ 3 & 3 & 0 & 1 & * & * \\ 3 & 3 & * & * & * & * \\ 0 & 1 & * & * & * & * \end{bmatrix}$$

The minimum number of linearly independent columns that $\mathcal{H}_{33}$ can have (ignoring the positions occupied by asterisks) is found to be 4. Thus, $\rho_M = 4$. The linearly independent columns are $\underline{h}_{11}$, $\underline{h}_{21}$, $\underline{h}_{12}$, and $\underline{h}_{22}$. This implies $\mu_1 = \mu_2 = 2$, and

$$L = [\underline{h}_{11} \quad \underline{h}_{12} \quad \underline{h}_{21} \quad \underline{h}_{22}]$$

From (2.23), $\sigma_{11} = \sigma_{12} = \sigma_{21} = \sigma_{22} = 2$; so, $\sigma_1 = \sigma_2 = 4$, $\sigma = 8$, and

$$M_1 = M_2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 3 & 1 & 2 \end{bmatrix}$$

The equations to solve for the coefficients $\{\gamma_{ijk}\}$ are

$$\hat{\underline{h}}_{13} = -M_1\underline{\gamma}_1$$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} \gamma_{111} \\ \gamma_{112} \\ \gamma_{121} \\ \gamma_{122} \end{bmatrix}$$

$$\hat{\underline{h}}_{23} = -M_2\underline{\gamma}_2$$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} \gamma_{211} \\ \gamma_{212} \\ \gamma_{221} \\ \gamma_{222} \end{bmatrix}$$

Since $\rho(M_1) = \rho(M_2) = 2$, there are 4 degrees of freedom in the choice of the eight parameters $\{\gamma_{ijk}\}$. Let γ_{111}, γ_{112}, γ_{221}, and γ_{222} be the arbitrary variables. Then the remaining ones are specified as

$$\gamma_{121} = 9 - \gamma_{111} \qquad \gamma_{122} = -3 - \gamma_{112}$$

$$\gamma_{211} = 8 - \gamma_{221} \qquad \gamma_{212} = -3 - \gamma_{222}$$

Define the characteristic polynomial of A as follows

$$|sI - A| = s^4 + \beta_4 s^3 + \beta_3 s^2 + \beta_2 s + \beta_1$$

Now substitute the $\{\gamma_{ijk}\}$ parameters in A according to (2.55), and evaluate its characteristic polynomial in terms of these parameters. Replace γ_{121}, γ_{122}, γ_{211}, and γ_{212} by their equivalent in terms of γ_{111} γ_{112}, γ_{221}, and γ_{222}. This results in a fourth-order polynomial involving these last four variables only. Equating the coefficients with those in the above definition of $|sI - A|$ gives the following matrix equation

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 9 & 0 \\ -3 & 8 & -3 & 9 \\ 1 & -3 & 1 & -3 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{111} \\ \gamma_{112} \\ \gamma_{221} \\ \gamma_{222} \end{bmatrix} + \begin{bmatrix} -72 \\ 51 \\ -9 \\ 0 \end{bmatrix}$$

It is observed that since the parameters γ_{111}, γ_{112}, γ_{221}, and γ_{222} are arbitrary and the matrix in the above equation is nonsingular, then the coefficients of the characteristic polynomial of A are arbitrary. Equivalently, there exists a unique set of parameters $\{\gamma_{ijk}\}$ that gives any four specified poles (subject to complex poles occuring in conjugate pairs) for a fourth-order minimal realization of the given sequence.

In terms of the coefficients of the characteristic polynomial of A, the parameters γ_{111}, γ_{112}, γ_{221}, and γ_{222} are specified as

$$\begin{bmatrix} \gamma_{111} \\ \gamma_{112} \\ \gamma_{221} \\ \gamma_{222} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 9 & 27 \\ 0 & -1 & -3 & 0 \\ 1 & 0 & -8 & -24 \\ 0 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} + \begin{bmatrix} 9 \\ 24 \\ 0 \\ -24 \end{bmatrix}$$

For the choice $\beta_1 = 4$, $\beta_2 = 10$, $\beta_3 = 10$, and $\beta_4 = 5$, the minimal partial realization is

$$A = \begin{bmatrix} 0 & -230 & 0 & -204 \\ 1 & 16 & 0 & 24 \\ 0 & 221 & 0 & 196 \\ 0 & -13 & 1 & -21 \end{bmatrix} \quad Q \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 \end{bmatrix}$$

and its matrix-fraction description is given by

$$\Phi(s) = \begin{bmatrix} 1 & 1 \\ s & s \end{bmatrix} \begin{bmatrix} s^2-16s+230 & -24s+204 \\ 13s-221 & s^2+21s-196 \end{bmatrix}^{-1}$$

The poles of this realization are at -1.0, -2.0, -1.0 $\pm$ j1.0 (as desired).

The following observations can be made. Notice that the only (minimal partial) realization invariants in this example are the controllability indices $\{\mu_1, \mu_2\}$ and the vectors $\{\underline{\ell}_{11}, \underline{\ell}_{12}, \underline{\ell}_{21}, \underline{\ell}_{22}\}$. None of the $\{\gamma_{ijk}\}$ parameters is an invariant in this problem. Notice also that the available degrees of freedom allow arbitrary poles for the minimal realization. In several applications (see Chapter 4), such a situation is desirable. The occurrence of this phenomenon is investigated in the following chapter. Finally, consider the following comparison with results obtained using the partial realization procedure of Ackermann (1972) to solve this problem.

In a recent technical note, Ledwich and Fortmann (1974) solved this example using the partial realization algorithm proposed by Ackermann (1972), and discussed some of the deficiencies of the algorithm. Their results show Ackermann's realization has dimension equal to 5 (not minimal) and four arbitrary poles, and with a reordering of the output terminals, the realization has dimension equal to 4 and three

arbitrary poles. If the dual of Ackermann's algorithm (as outlined in Remark (2.83)) is used, the realization has dimension equal to four and two arbitrary poles for both possible orderings of the input terminals.

Ledwich and Fortmann (1974) also point out that the choice of the arbitrary parameters and the ordering of the input (output) terminals in Ackermann's algorithm can increase the dimension of the realization. These deficiencies are a consequence of the procedure used to obtain the linearly independent columns of $\mathcal{H}_{N_0 N_0}$, because the procedure does not account for the possibility that $\underline{\hat{h}}_{jk}$ may not belong to the basis if it is compared with the portions of the same length of $\underline{\hat{h}}_{ik'}$ for $i = 1,2,\ldots,r$ and $k' \leq k$, rather than just for $i \leq j$ and $k' \leq k$. This search procedure (generally) leads to the identification of less than σ + pm realization parameters, but it is clear by now that a considerable loss in generality (and possibly even in minimality) is also expected to result. ∇

In closing this chapter, it remains to state that the information about the number and character of the available degrees of freedom and about the structure of the minimal realization which is obtained using Algorithm (2.78), can also be obtained (at least conceptually) via other realization algorithms (for example, Ho and Kalman, 1966; Tether, 1970; Kalman, 1971b; Rissanen, 1971; Silverman, 1971), but the amount of effort (and bookkeeping) involved is discouraging.

CHAPTER 3
NEW PROBLEMS IN PARTIAL REALIZATION THEORY

Often it is desirable (possibly necessary) that the partial realization obtained for a specified finite sequence be stable besides having dimension as small as possible. The situation also occurs where the realization is required to have arbitrary poles in addition to having dimension as small as possible. These problems arise in network theory, where a linear, time-invariant, stable network must often be synthesized to exhibit a (partially) prescribed impulse response. They also arise in control theory in the design of minimal order observers, where they are problems of significant importance. Besides having these relevant applications, these two problems are of interest in their own right.

The purpose of this chapter is to formulate the minimal partial stable realization problem and the minimal partial arbitrary realization problem, and to present a solution in the form of realization algorithms. In other words, the construction of extension sequences that are not necessarily minimal but have other properties (namely, stable or arbitrary poles) is considered here. It is interesting to note that these problems have not been discussed elsewhere.

Problem Statement

The minimal partial stable and the minimal partial arbitrary realization problems are defined next.

(3.1) Definition. Given a finite sequence $\{L_1, L_2, \ldots, L_{N_0}\}$ of $m \times r$ matrices which define the first N_0 terms in the impulse response of a (possibly infinite-dimensional)

system, find a triple $(A,Q,D)_p$ such that

a) $L_i = DA^{i-1}Q$ $\qquad i = 1,2,\ldots,N_0$

b) p is as small as possible, and

c) A has all its eigenvalues with negative real parts.

The triple $(A,Q,D)_p$ is then said to be a minimal partial stable realization of $\{L_1,L_2,\ldots,L_{N_0}\}$.

(3.2) Definition. Given a finite sequence $\{L_1,L_2,\ldots,L_{N_0}\}$ of $m\times r$ matrices which define the first N_0 terms in the impulse response of a (possibly infinite-dimensional) system, find a triple $(A,Q,D)_p$ such that

a) $L_i = DA^{i-1}Q$ $\qquad i = 1,2,\ldots,N_0$

b) p is as small as possible, and

c) A has arbitrary eigenvalues.

The triple $(A,Q,D)_p$ is then said to be a minimal partial arbitrary realization of $\{L_1,L_2,\ldots,L_{N_0}\}$.

(3.3) Definition. The value of p which satisfies Definition (3.1) is denoted as p_S, and p_A denotes the value of p which satisfies Definition (3.2).

The solution to the above-defined problems requires the construction of extension sequences which may not be minimal but possess other properties; namely, stable or arbitrary poles. These extension sequences are referred to as minimal stable and minimal arbitrary extension sequences, respectively.

In the preceding chapter, a procedure due to Kalman (1971b) and Tether (1970) to construct a minimal extension sequence is briefly discussed. With a slight modification, the same approach can also be

used to construct an extension sequence which is not minimal. Namely, the elements of the extension sequence are now chosen to disrupt (as opposed to preserve) existing linear dependencies among the columns of $\mathcal{H}_{N_0 N_0}$ in (2.11). For a given partial sequence, there exist infinitely many ways of doing so. But one must again be extremely careful, because a poorly chosen extension sequence can force the minimal realization of the complete (partial plus extension) sequence to have an undesirably large dimension (possibly infinite).

The problem of constructing an extension sequence becomes much more complicated if the extension sequence is to be such that the minimal realization of the complete (partial plus extension) sequence must have a prescribed dimension and/or other properties, as in the problems of Definitions (3.1) and (3.2). An attractive and reasonable way to attack these problems is to investigate the basic structure and parameters common to all partial realizations having the desired properties, as done in Chapter 2 for minimal partial realizations. Such a solution is presented in this chapter for the minimal partial stable and the minimal partial arbitrary realization problems.

The approach taken is to first show the existence of arbitrary extension sequences for a partial sequence. It follows that there is a minimal one among all the arbitrary extension sequences. The existence of stable extension sequences (and, correspondingly, of a minimal one) is also implied (in view of the direction of the inequalities in (3.4)). The proofs of these results suggest iterative-type algorithms to solve the corresponding realization problems.

Existence of Stable and Arbitrary Extension Sequences

In view of Example (2.85), it is clear that stable and arbitrary extension sequences do exist, at least for some partial sequences. That such extension sequences do indeed exist for all partial sequences, and the relation of p_S and p_A with p_M is shown in this section.

Based on the results of Chapter 2 it can be stated that all minimal partial realizations of a given finite sequence have the same set of controllability indices, and that a partial realization of a given finite sequence cannot have dimension lower than p_M. It follows that

(3.4) $$p_A \geqq p_S \geqq p_M$$

and that the controllability indices of a partial realization cannot be smaller than the controllability indices of a minimal partial realization (of the same finite sequence). This is a very important point, and it should be kept in mind throughout the remainder of this dissertation. It is made more precise with the aid of the following definitions.

(3.5) Definition. The set of controllability indices of a minimal partial realization of a finite matrix sequence is called the set of minimal controllability indices and it is denoted as $\{\mu_{iM}\}$.

(3.6) Definition. A set of indices $\{\mu_i\}$ is said to be an admissible set of controllability indices for a partial realization of a finite matrix sequence if and only if

(3.7) $$\mu_i \geqq \mu_{iM} \qquad i = 1,2,\ldots,r$$

where $\{\mu_{iM}\}$ is the set of minimal controllability indices of the given partial sequence.

(3.8) Definition. The set of all admissible sets of controllability indices for a partial realization of dimension p of a given finite matrix sequence is denoted by $\mathcal{M}(p)$. That is,

$$\mathcal{M}(p) = \{\{\mu_i\} : \{\mu_i\} \text{ is an admissible set of controllability indices and } \mu_1 + \mu_2 + \ldots + \mu_r = p\}$$

In the light of these definitions, if a finite matrix sequence admits a partial (non-minimal) realization of dimension p, then the controllability indices of the realization belong to $\mathcal{M}(p)$. However, it is not necessarily true that the elements of $\mathcal{M}(p)$ for a given finite matrix sequence and a specified value for p are the controllability indices of a partial realization of dimension p of the given matrix sequence. In other words, it is possible that a finite matrix sequence does not admit any partial realization of a given dimension. This is discussed in more detail in the sequel (Remark (3.35)).

(3.9) Notation. In the remainder of this dissertation, the characteristic polynomial of matrix A is denoted as

$$\beta(s) = s^p + \beta_p s^{p-1} + \ldots + \beta_2 s + \beta_1$$

and the set of coefficients $\{\beta_k\}$ defines the following p-dimensional vector

$$\underline{\beta}^T = [\beta_1 \quad \beta_2 \quad \ldots \beta_p] \qquad \nabla$$

The requirements that $(A,Q,D)_p$ define a stable system or a system with arbitrary poles can now be equivalently replaced by the requirements that the coefficients $\{\beta_k\}$ define a stable polynomial (that is, a polynomial whose roots are all in the left-hand of the s-plane) or be arbitrary, respectively.

The various results that follow state several necessary and/or sufficient conditions for a partial stable or arbitrary realization to exist, and constitute the basis for the realization algorithms presented in the next section.

(3.10) Theorem. A triple $(A,\underline{q},D)_p$ is a partial arbitrary realization of a finite sequence $\{\underline{\ell}_1,\underline{\ell}_2,\ldots,\underline{\ell}_{N_0}\}$ of m-dimensional (column) vectors if and only if $p \geqq N_0$.

Proof. Sufficiency. Notice that for $r = 1$ there is only one controllability index and it is equal to p. Also, as shown in the proof of Algorithm (2.78), $\mu \geqq N_0$ is indeed an admissible controllability index.

From the proof of Lemma (2.21) it can be concluded that the $\{\gamma_{ijk}\}$ coefficients are equal to the $\{\beta_k\}$ coefficients in this case. Therefore, (2.32) becomes

(3.11) $$\underline{\ell}_{p+1+\tau} = -\sum_{k=1}^{p} \beta_k \underline{\ell}_{k+\tau} \qquad \tau = 0,1,2,\ldots$$

For $p \geqq N_0$ the left-hand side of (3.11) is unspecified. Thus, an extension sequence $\{\underline{\ell}_{N_0+1},\underline{\ell}_{N_0+2},\ldots\}$ can be chosen to satisfy (3.11) for any given set of coefficients $\{\beta_k\}$. The triple $(A,\underline{q},D)_p$ can be chosen in the single-input form of (2.55)-(2.57):

(3.12) $$A = \left[\begin{array}{c|c} \underline{0}^T_{p-1} & \\ \hline I_{p-1} & -\underline{\beta} \end{array}\right]$$

(3.13) $$\underline{q} = \underline{i}_1$$

(3.14a) $$D = [\underline{d}_1 \quad \underline{d}_2 \quad \cdots \quad \underline{d}_p]$$

(3.14b) $$\underline{d}_i = \underline{\ell}_i \qquad i = 1,2,\ldots,N_0$$

Necessity. Let $p \geqq N_0 - 1$. For $\tau = 0,1,\ldots,N_0-p-1$, equation (3.11) can be written as

$$(3.15)\qquad \begin{bmatrix} \underline{\ell}_{p+1} \\ \underline{\ell}_{p+2} \\ \cdot \\ \cdot \\ \cdot \\ \underline{\ell}_{N_0} \end{bmatrix} = - \begin{bmatrix} \underline{\ell}_1 & \underline{\ell}_2 & \cdots & \underline{\ell}_p \\ \underline{\ell}_2 & \underline{\ell}_3 & \cdots & \underline{\ell}_{p+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \underline{\ell}_{N_0-p} & \underline{\ell}_{N_0-p+1} & \cdots & \underline{\ell}_{N_0-1} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_p \end{bmatrix}$$

This is a set of $m(N_0-p)$ equations in p unknowns. A completely arbitrary solution vector exists if and only if the nullity of the matrix in (3.15) is equal to p. Such a condition is impossible unless the sequence is a null sequence (that is, all its elements are zero), and as stated in the discussion which follows Remark (2.16), this trivial case is excluded. ∇

(3.16) Notation. The symbol $\underline{\zeta}_i^T$ is used in the sequel to represent the ith element of a sequence of r-dimensional row vectors. This new symbol is preferred over the symbol $\underline{\ell}_i^T$ to avoid confusing a row vector sequence with the transpose of a column vector sequence. Likewise, to avoid confusing the rows of L_k with the transpose of the columns of L_k, the symbol $\underline{\zeta}_{jk}^T$, rather than $\underline{\ell}_{jk}^T$, is used to represent the jth row of L_k. ∇

The dual of Theorem (3.10) is now stated.

(3.17) Theorem. A triple $(A,Q,\underline{d})_p$ is a partial arbitrary realization of a finite sequence $\{\underline{\zeta}_1^T,\underline{\zeta}_2^T,\ldots,\underline{\zeta}_{N_0}^T\}$ of r-dimensional (row) vectors if and only if $p \geq N_0$.

Proof. Follows by duality. ∇

Of all the possible partial arbitrary realizations, that one which is minimal is the one of greatest importance here. To this end, the results of Theorems (3.10) and (3.17) are combined into the following corollary.

(3.18) Corollary. The dimension of the minimal partial arbitrary realization of a (column or row) vector sequence of N_0 elements is given by N_0.

Proof. Follows readily from the proofs of Theorems (3.10) and (3.17). √

Notice that Theorems (3.10) and (3.17), as well as Corollary (3.18), include the scalar sequence case ($m = r = 1$) as a special case.

A result equivalent to that of Corollary (3.18) for finite matrix sequences seems to be more difficult to obtain. This is probably due to the fact that in the multiple input/output case the coefficients of the characteristic polynomial of A are no longer equal to the $\{\gamma_{ijk}\}$ coefficients but are defined by these through nonlinear equations, and the $\{\gamma_{ijk}\}$ coefficients are determined, in turn, by the elements of the matrix sequence. These sets of relations are not easily expressed in closed form. However, necessary conditions and sufficient conditions for a triple $(A,Q,D)_p$ to be a partial arbitrary realization of a finite matrix sequence can be stated. These are given in the next two theorems.

(3.19) Theorem. Necessary conditions for a triple $(A,Q,D)_p$ to be a partial arbitrary realization of a finite sequence $\{L_1, L_2, \ldots, L_{N_0}\}$ of $m \times r$ matrices are that

a) the controllability indices of $(A,Q,D)_p$ be an admissible set of controllability indices, and

b) $$p \leq \sum_{i=1}^{r} \sum_{j=1}^{r} \sigma_{ij} - \sum_{i=1}^{r} \rho(\hat{M}_i) = \sigma - \sum_{i=1}^{r} \rho(\hat{M}_i)$$

where the integers $\{\sigma_{ij}\}$ are defined by (2.23) and $\hat{M}_i$ is defined in Step 2 of Algorithm (2.78).

Proof. The necessity of condition (3.19a) is obvious. Condition (3.19b) is proved as follows.

Generally, the $\{\beta_k\}$ coefficients are nonlinear functions of the $\{\gamma_{ijk}\}$ coefficients. Therefore, in order for A to have arbitrary eigenvalues it is necessary that at least p of the $\{\gamma_{ijk}\}$ coefficients be arbitrary. To obtain the $\{\gamma_{ijk}\}$ coefficients of a partial realization of the given sequence, it is necessary to solve an equation of the form (2.79) for i = 1,2,...,r. The total number of degrees of freedom (arbitrary parameters) in the solution to these equations was found to be given by (2.80) with $\rho(\hat{M}_i) = 0$ if $\mu_i \geqq N_0$. For the realization to have completely arbitrary poles, it is necessary that p be less than or equal to this number. This is condition (3.19b). ∇

(3.20) Remark. Dual necessary conditions for a matrix triple to be a partial arbitrary realization (of a given finite sequence) can be obtained by considering the dual arguments in the proof of Theorem (3.19). In Definition (2.19) and in Remark (2.33), the dual of the sets of invariants $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ and of the parameters $\{\sigma_{ij}\}$ were defined to be $\{\delta_i\}$, $\{\theta_{ijk}\}$, and $\{\kappa_{ij}\}$, respectively. Correspondingly, the dual of the $\{\underline{\ell}_{jk}\}$ (column) vector invariants are rows of the elements in the given matrix sequence. As previously stated (see (3.16)) these dual invariants will be denoted by the set of (row) vectors $\{\underline{\zeta}_{jk}^T\}$.

Now let $\{\delta_{iM}\}$ denote the set of minimal observability indices, and define an admissible set of observability indices as $\{\delta_i : \delta_i \geqq \delta_{iM}\}$, analogous to Definitions (3.5) and (3.6). Also, define the integer κ as

$$\kappa = \sum_{i=1}^{m} \kappa_i = \sum_{i=1}^{m} \sum_{j=1}^{m} \kappa_{ij} \tag{3.21}$$

and let $\check{R}_i$ denote the matrix corresponding to $\hat{M}_i$ in the dual formulation of (2.79). With these definitions, the dual of the necessary conditions (3.19a) and (3.19b) are given as

a) the observability indices of $(A,Q,D)_p$ be an admissible set of observability indices, and

b) $p \leqq \kappa - \sum_{i=1}^{m} \rho(\hat{R}_i)$

Using either (3.19b) or (3.20b) to obtain lower bounds for the dimension of the minimal partial arbitrary realization of a given matrix sequence should give equivalent results. However, this seems to be difficult to prove. In the sequel, reference is made mostly to (3.19b), but it should be kept in mind that (3.20b) is equally applicable. ∇

It is stressed that Theorem (3.19) gives only necessary conditions for a matrix triple to be a partial arbitrary realization (of a given finite sequence) of dimension equal to p. Equation (3.19b) being satisfied for a value of p and an admissible set $\{\mu_i\}$ does not imply that a partial realization of dimension p and structure determined by the $\{\mu_i\}$ is a partial realization of the given finite sequence. This is due to the fact that p or more arbitrary γ_{ijk} coefficients do not (generally) guarantee that the p eigenvalues of A are arbitrary. The theorem does give the lowest possible value of p and the corresponding admissible sets of controllability indices (notice that, for a given value of p, equation (3.19b) may be satisfied with more than one admissible set of controllability indices) that are suitable candidates for the dimension and controllability indices of a minimal partial arbitrary realization of the given sequence.

The following result states sufficient conditions for a matrix triple to be a partial arbitrary realization of a finite matrix sequence.

(3.22) Theorem. Sufficient conditions for a triple $(A,Q,D)_p$ to be a partial arbitrary realization of a finite sequence $\{L_1,L_2,\ldots,L_{N_0}\}$ of m×r matrices are that

a) the controllability indices of $(A,Q,D)_p$ be an admissible set of controllability indices, and

b) $p \geqq \min(rN_0, mN_0)$

Proof. The proof is by construction. Let the structure of a matrix triple be determined by the following set of controllability indices $\{\mu_i : \mu_i \geqq N_0\}$. This is an admissible set of controllability indices because the size of the minimal controllability indices is, at most, equal to N_0. Clearly, $p \geqq rN_0$ for such a choice of input structure. Consider now the equations (2.32) for a partial realization of the given sequence that has the selected input structure. It is noticed that since $\mu_i + 1 + \tau > N_0$ for all $\tau \geqq 0$ and $1 \leqq i \leqq r$, then all the vectors on the left-hand side of (2.32) and (possibly) some vectors on the right-hand side of (2.32) are unspecified. Therefore, the $\{\gamma_{ijk}\}$ coefficients can be chosen at will, and consequently, the eigenvalues of A are arbitrary.

If the above procedure is applied in dual context, it is found that a matrix sequence of N_0 terms always admits a partial realization of dimension $p \geqq mN_0$ and arbitrary poles. Equation (3.22b) follows. ∇

The results of Theorems (3.19) and (3.22) are easily combined to obtain a set of constraints that must be satisfied by a matrix triple in order to be a minimal partial arbitrary realization of a given finite matrix sequence. This is the following corollary.

(3.23) Corollary. If a triple $(A,Q,D)_{p_A}$ is a minimal partial arbitrary realization of a finite sequence $\{L_1, L_2, \ldots, L_{N_0}\}$ of $m \times r$ matrices, then its dimension and its controllability indices $\{\mu_i\}$ satisfy the following constraints:

a) $\mu_i \geqq \mu_{iM}$ $\qquad i = 1,2,\ldots,r$

b) $p_A \leqq \sigma - \sum_{i=1}^{r} \rho(\hat{M}_i)$

c) $p_A \leqq \min(rN_0, mN_0)$

where $\{\mu_{iM}\}$ is the set of minimal controllability indices associated with the given sequence, σ is defined by (2.51) and (2.23), and $\hat{M}_i$ is defined in Step 2 of Algorithm (2.78).

Proof. Follows readily from the proofs of Theorems (3.19) and (3.22). ∇

(3.24) Remark. It is easy to show that constraints (3.23b) and (3.23c) are equal to each other, and in turn, equal to N_0 whenever $r = 1$ and/or $m = 1$, and $\mu = N_0$ is the chosen admissible controllability index. Thus, Corollary (3.18) is a special case of Corollary (3.23). ∇

Corollary (3.23) is more significant than what appears at first glance. It gives upper and lower bounds for p_A and all the possible corresponding sets of admissible controllability indices. (Constraint (3.23c) is obviously an upper bound, and even though (3.23b) looks like an upper bound also, it is actually a lower bound; this is discussed in length later on.) Notice that constraints (3.23a) and (3.23c) are easily verified for a given matrix triple and a given matrix sequence, but (3.23b) requires (in general) the computation of the rank of r matrices. The minimal partial arbitrary realization algorithm that appears in the next section is based on the constraints specified in the above corollary.

These results show that every finite matrix sequence admits a minimal arbitrary extension sequence, and give bounds for the dimension of the minimal partial arbitrary realization. It follows that every finite matrix sequence admits infinitely many extension sequences and has infinitely many partial stable realizations (construct any partial arbitrary realization and select stable poles for it). Then there exists at least one realization of minimal dimension among all the partial stable realizations. This minimal realization (or realization, if not unique) is the one of primary importance in this dissertation.

At this point, it should be intuitively clear that a partial stable realization which is obtained by specifying stable poles in a partial arbitrary realization is not (generally) minimal; the minimal partial stable realization is expected to have dimension larger than p_M but smaller than p_A. Given a finite matrix sequence, it is not too difficult to calculate the value of p_M (see (2.12)) and (strong bounds for) the value of p_A (see Corollary (3.23)). Unfortunately, it seems to be quite difficult to state analogous results for the value of p_S. This is probably due to the fact that the stability of a realization is determined exclusively by the factors of the characteristic polynomial of A (the eigenvalues of A), and the elements of A (assuming A to be in the form (2.55)-(2.57)) that determine its characteristic polynomial are specified by the elements of the given sequence via formulas that are not easily expressed in closed form. Further, stability is normally determined from the characteristic polynomial coefficients using the well-known Routh-Hurwitz criterion (see, for example, Gantmacher, 1959, Vol. 2), and this procedure relates the coefficients in a fashion that is not easily expressed in closed form either. In other words, even

though there is no doubt that every finite matrix sequence has a minimal partial stable realization, it seems to be difficult to state a result for the value of p_S as strong as Corollary (3.23) for p_A.

On the other hand, it is possible to state some simple necessary conditons that must be satisfied by a system (2.1) in order to be a partial stable realization of a finite matrix sequence. These conditions are stated and proved for the vector sequence case and then for the matrix sequence case. But first it is convenient to define the following matrices. Let B_p be the following mr×p matrix

$$B_p = \begin{bmatrix} \underline{\ell}_{11} & \underline{\ell}_{12} & \cdots & \underline{\ell}_{1p} \\ \underline{\ell}_{21} & \underline{\ell}_{22} & \cdots & \underline{\ell}_{2p} \\ \vdots & \vdots & & \vdots \\ \underline{\ell}_{r1} & \underline{\ell}_{r2} & \cdots & \underline{\ell}_{rp} \end{bmatrix} = \begin{bmatrix} \underline{b}_{p1}^T \\ \underline{b}_{p2}^T \\ \vdots \\ \underline{b}_{pr}^T \end{bmatrix} \tag{3.25}$$

and let G_p be the following p×mr matrix

$$G_p = \begin{bmatrix} \underline{\zeta}_{11}^T & \underline{\zeta}_{21}^T & \cdots & \underline{\zeta}_{m1}^T \\ \underline{\zeta}_{12}^T & \underline{\zeta}_{22}^T & \cdots & \underline{\zeta}_{m2}^T \\ \vdots & \vdots & & \vdots \\ \underline{\zeta}_{1p}^T & \underline{\zeta}_{2p}^T & \cdots & \underline{\zeta}_{mp}^T \end{bmatrix} = [\underline{g}_{p1} \quad \underline{g}_{p2} \quad \cdots \quad \underline{g}_{pm}] \tag{3.26}$$

where $\underline{\ell}_{jk}$ and $\underline{\zeta}_{jk}^T$ are as previously defined. When r = 1, B_p becomes $\underline{b}_p^T$, a single (block) row; likewise, when m = 1, G_p becomes $\underline{g}_p$, a single (block) column.

(3.27) Theorem. Necessary conditions for a triple $(A,\underline{q},D)_p$ with $p_M \leqq p \leqq N_0 - 1$ to be a partial stable realization of a finite sequence $\{\underline{\ell}_1, \underline{\ell}_2, \ldots, \underline{\ell}_{N_0}\}$ of m-dimensional

(column) vectors are that

a) $\rho(\underline{b}_p^T) = \rho(\underline{b}_{p+1}^T)$, and

b) every non-zero row of $\underline{b}_{p+1}^T$ have two or more elements with different sign.

Proof. The bounds on p follow from the definition of p_M and from Theorem (3.10). As in the proof of Theorem (3.10), equation (2.32) reduces to (3.11) when r = 1. This equation must be satisfied (for some $\{\beta_k\}$) for the sequence to admit a partial realization of dimension p. Notice that condition (3.27a) is another form for (3.11) with $\tau = 0$. This proves necessity of (3.27a).

Suppose now that (3.27a) is satisfied for some value of p in the specified range. The scalars $\{\beta_k\}$ that determine the relation (3.11) are the coefficients of the characteristic polynomial of A. In order for A to have all its eigenvalues in the left-half of the s-plane, it is necessary that all the $\{\beta_k\}$ coefficients be positive (this is a well-known result in the theory of polynomials). It is easy to see that at least one of these coefficients is negative if all the elements in any non-zero row of $\underline{b}_{p+1}^T$ have the same sign. This proves necessity of (3.27b). ∇

The dual of the above result is now stated.

(3.28) Theorem. Necessary conditions for a triple $(A,Q,\underline{d})_p$ with $p_M \leq p \leq N_0 - 1$ to be a partial stable realization of a finite sequence $\{\underline{\zeta}_1^T, \underline{\zeta}_2^T, \ldots, \underline{\zeta}_{N_0}^T\}$ of r-dimensional (row) vectors are that

a) $\rho(\underline{q}_p) = \rho(\underline{q}_{p+1})$, and

b) every non-zero column of $\underline{q}_{p+1}$ have two or more elements with different sign.

Proof. Follows by duality. ∇

Notice that Theorems (3.27) and (3.28) include the scalar sequence case ($m = r = 1$) as a special case.

Necessary conditions analogous to those given above can be obtained for the matrix sequence case with the aid of the following lemma due to Ho and Kalman (1966). The simple proof is omitted.

(3.29) Lemma. The elements of an infinite sequence $\{L_1, L_2, \ldots\}$ of $m \times r$ matrices which admits a finite-dimensional realization (2.1) satisfy the following relation

(3.30) $$L_{p+1+\tau} = -\sum_{k=1}^{p} \beta_k L_{k+\tau} \qquad \tau = 0,1,2,\ldots$$

Further, if the degree of the minimal polynomial of A is smaller than p, then (3.30) is also satisfied with p replaced by the degree of the minimal polynomial of A and the $\{\beta_k\}$ replaced by the coefficients of the minimal polynomial of A.

Now the matrix sequence version of Theorems (3.27) and (3.28) can be stated.

(3.31) Theorem. Necessary conditons for a triple $(A,Q,D)_p$ with $p \geqq p_M$ to be a partial stable realization of a finite sequence $\{L_1, L_2, \ldots, L_{N_0}\}$ of $m \times r$ matrices are that

a) $\rho(B_p) = \rho(B_{p+1})$, and

b) every non-zero row of B_{p+1} have two or more elements with different sign.

Proof. Follows the pattern of the proof of Theorem (3.27) except that (3.30) is used rather than (3.11). ∇

Theorem (3.31) also has a dual result; its form should be obvious and is not given here. For brevity, the dual approach is not considered any further in this chapter.

Notice that Theorem (3.27) is a special case of Theorem (3.31). However, a situation can arise in the partial matrix sequence case that never occurs in the partial vector sequence case. This is discussed after the following definition.

(3.32) Definition. Matrix A is said to be cyclic if and only if any one of the following equivalent conditions is satisfied.

a) The minimal polynomial and the characteristic polynomial of A are equal to each other.

b) The Jordan form of A (see, for example, Gantmacher, 1959, Vol. 1) has only one Jordan block corresponding to every distinct eigenvalue of A.

c) There exists a p-dimensional vector $\underline{q}$ such that $(A,\underline{q})_p$ is a completely controllable pair.

By a slight abuse of notation, system (2.1) is also called cyclic if matrix A is cyclic.

In the analysis and design of systems of the form (2.1), cyclicity is a very convenient (but by no means necessary) property. The same holds true in realization theory.

(3.33) Remark. It follows from (3.32c) that a completely controllable single input system is always cyclic. As a consequence, it is true that

$\rho(\underline{b}_k^T) = \rho(\underline{b}_{k+1}^T)$ if and only if $k \geq p_M$ in the partial (column) vector sequence case. When the system has more than one input, it is no longer true that complete controllability implies cyclicity. Consequently, in the partial matrix sequence case $k \geqq p_M$ still implies $\rho(B_k) = \rho(B_{k+1})$, but $\rho(B_k) = \rho(B_{k+1})$ no longer implies $k \geqq p_M$ (if condition (3.32a) is not true, then (3.30) is satisfied for some $p < p_M$). Simon and Mitter (1968) have discussed extensively the question of complete controllability of cyclic and noncyclic systems.

The above discussion involves a subtle but significant point: the fact that complete controllability does not imply cyclicity in the multiple input case renders condition (3.31a) less useful than conditon (3.27a) in the single input case. More specifically, given a finte (column) vector sequence, p_M is equal to the smallest integer k such that $\rho(\underline{b}_k^T) = \rho(\underline{b}_{k+1}^T)$, but if a finite matrix sequence is given instead, then p_M is not generally equal to the smallest integer k such that $\rho(B_k) = \rho(B_{k+1})$. ∇

(3.34) Remark. The results presented in this section can be used to state several conditions under which equality or inequality holds in (3.4). For example, if $p_M = \min(rN_0, mN_0)$, then $p_A = p_S = p_M$; or, if the minimal partial realization is unstable and $p_M + 1 = \min(rN_0, mN_0)$, then $p_A = p_S = p_M + 1$. More of these conditions could be stated, but it is not enlightening because of their data-dependent nature. In other words, everything centers on the value of p_M and the number of arbitrary parameters in the minimal partial realization, and these are determined exclusively by the elements of the given sequence. Further, all that is needed to recognize these special cases is some familiarity with the results stated in Corollary (3.23) and in Theorem (3.31). ∇

Partial Realization Algorithms

The theory presented above and in Chapter 2 is used here to develop procedures to solve the minimal partial stable and the minimal partial arbitrary realization problems. Unlike Algorithm (2.78), the algorithms presented here are of iterative nature. This is a consequence of the difficulties encountered in relating the elements of a finite matrix sequence to the coefficients of a stable or arbitrary polynomial of a specific degree, and the host of possible situations that arise in the construction of non-minimal partial realizations (see Remark (3.35) below).

When the given partial sequence is composed of vectors rather than matrices, it is considerably simpler to obtain the minimal partial stable and the minimal partial arbitrary realizations. But the algorithms are discussed for the matrix case only since the simplifications that are possible in the vector case are obvious.

(3.35) Remark. As mentioned above, the construction of non-minimal partial realizations of a specified dimension and other desirable properties is not any easy problem. Several difficulties that do not arise in the construction of minimal partial realizations are intrinsic problems in the construction of non-minimal partial realizations.

Recall that all minimal realizations of a partial matrix sequence have the same controllability indices and are completely controllable and observable. Such is not necessarily the case with non-minimal partial realizations. There (generally) exist non-minimal partial realizations of the same dimension but with different sets of controllability indices. Also, it is possible that a completely controllable non-minimal

partial realization is unobservable. In fact, the following situations (among others) can occur in the non-minimal partial realization problem.

a) Some realizations with controllability indices $\{\mu_i\} \in M(p)$ are completely observable while other realizations with the same controllability indices are unobservable.

b) All (or some) realizations with controllability indices $\{\mu_i\}_j \in M(p)$ are completely observable while all (or some) realizations with controllability indices $\{\mu_i\}_k \in M(p)$ for $j \neq k$, are unobservable.

c) All realizations of dimension p are unobservable.

Here $\{\mu_i\}_j$ denotes the jth element of $M(p)$; also, it is assumed that $p > p_M + 1$. The occurrence of (3.35a)-(3.35c) is explained next.

Suppose that a finite matrix sequence is given, together with its minimal realization (of dimension p_M). For the moment, assume the minimal realization is unique. Now obtain the corresponding minimal extension sequence using (2.32). Suppose now that one or more of the elements of the minimal extension sequence is altered in a random fashion. Since the minimal extension sequence is unique, such an action forces the dimension of the minimal realization of the modified (complete) sequence to be larger than p_M. This is a consequence of the disruption of existing linear dependencies among the columns of $\mathcal{H}$. The first p linearly independent columns of $\mathcal{H}_{N_0 N_0}$ determine the minimal controllability indices, so the least increase in dimension should occur when only the first elements of the minimal extension sequence are altered. Furthermore, whether the increase in dimension is large or small depends on the type and amount of alterations done to the minimal extension sequence,

and it is conceivable (and indeed true) that for some sequences even the slightest changes done to the first few elements of the minimal extension sequence cause an increase in dimension of 2 or more (from p_M to $p_M + k$, $k \geq 2$). When the minimal realization of the given finite sequence is not unique, the minimal extension sequence is not unique. The preceding arguments also apply to this case with the observation that some specific alterations done to any particular minimal extension sequence do not increase the dimension of the minimal realization of the modified (complete) sequence.

The occurrence of (3.35a)-(3.35c) can also be explained from the point of view of the structural parameters of a non-minimal realization. It has been shown that a non-minimal realization has more arbitrary parameters than a minimal realization of the same sequence. Also, as stated in Remark (2.58), the parameters $\{\gamma_{ijk}\}$ in A and the vectors $\{\underline{\ell}_{jk}\}$ which comprise the columns of D must jointly satisfy the condition that $(A,D)_p$ be a completely observable pair. When $p = p_M$, the realization will always be observable for all possible values of all the arbitrary γ_{ijk} parameters; but when $p > p_M$, the realization can be unobservable for some or all possible values of all the arbitrary γ_{ijk} parameters. ∇

Algorithm (2.78) gives all the possible minimal realizations of a partial matrix sequence at once (solving equations (2.79) one time only), because the set of minimal controllability indices is unique. But to obtain all possible non-minimal realizations of specified dimension of a partial matrix sequence (generally) requires solving equations of the form (2.79) several times, and it is also necessary to verify whether the realizations are observable or not. Likewise, to obtain all possible stable or arbitrary realizations of a partial matrix sequence it is

(generally) necessary to solve several sets of equations of the form (2.79), verify the observability of the realizations, and further, the stability or arbitrariness of the realizations also has to be verified. These are the basic ideas involved in the partial realization algorithms presented in the sequel.

Before discussing the minimal partial stable and arbitrary realization algorithms, a non-minimal partial realization algorithm that can be used to obtain all the possible realizations of specified dimension $p > p_M$ of a given finite matrix sequence is presented. This algorithm is the most important part of each iteration in the minimal partial stable and minimal partial arbitrary realization algorithms, besides being of interest in its own right. It is assumed that the minimal set of controllability indices (and, consequently, $\mathcal{M}(p)$) is known and given as an input to the algorithm.

(3.36) Non-minimal Partial Realization Algorithm.

Step 1. Form the truncated generalized Hankel matrix $\mathcal{H}_{N_0 N_0}$ (the upper left-hand corner of (2.11)). Choose one set of admissible controllability indices in $\mathcal{M}(p)$.

Step 2. Form the following $mN_0 \times p$ matrix

(3.37) $$L = [\underline{h}_{11} \quad \underline{h}_{12} \cdots \underline{h}_{1\mu_1} \quad \underline{h}_{21} \cdots \underline{h}_{2\mu_2} \cdots \underline{h}_{r1} \cdots \underline{h}_{r\mu_r}]$$

and define M_i as the $m(N_0 - \mu_i) \times p$ top portion of L.

Step 3. According to the desired input structure,

(3.38) $$\underline{\hat{h}}_{i\mu_i+1} = -\hat{M}_i \underline{\gamma}_i \qquad i = 1,2,\ldots,r$$

where the hat (^) over M_i indicates that some columns of M_i are not generally included, and $\underline{\hat{h}}_{i\mu_i+1}$ is the $m(N_0 - \mu_i)$ top portion of $\underline{h}_{i\mu_i+1}$. Solve (3.38) for $i = 1,2,\ldots,r$ to obtain

the realization parameters $\{\gamma_{ijk}\}$. Notice that there will always be some arbitrary γ_{ijk} because increasing one or more of the minimal controllability indices forces equation (3.38) to have more unknowns than equations for one or more values of i. That is, as μ_{jM} is increased to $\mu_{jM} + 1$, $\hat{M}_j$ has m less rows while the number of unknowns has increased by one (at least for i = j). In the extreme, $\mu_i \geq N_0$ makes M_i be undefined and, therefore, $\underline{\gamma}_i$ is arbitrary.

Step 4. A partial realization of dimension p can be constructed in the form (2.55)-(2.57) with the $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ parameters of Steps 1 and 3, and the p m-dimensional vectors that make up the first block row of L.

Step 5. Form the m(p-m+1)×p observability matrix

$$\hat{O} = \begin{bmatrix} D \\ DA \\ \cdot \\ \cdot \\ \cdot \\ DA^{p-m} \end{bmatrix} \tag{3.39}$$

and compute its rank. The realization is completely observable if and only if $\rho(\hat{O}) = p$. In (3.39) it has been implicitly assumed that D has full rank. If such is not the case, then $\hat{O}$ has to be extended to include up to the p - 1 power of A.

Step 6. Repeat Steps 2-5 for a different set of admissible controllability indices in $M(p)$. This is continued until all the elements of $M(p)$ are considered.

Proof. This algorithm is closely related to Algorithm (2.78), and the proof follows that of Algorithm (2.78) with two modifications to account

for the differences that exist between the minimal and non-minimal partial realization problems (see Remark (3.35)).

The modifications are the following. First, since it is possible that there exist several non-minimal realizations of the same dimension but with different sets of controllability indices, then every admissible set of controllability indices that add up to p could possibly define the structure of a valid (controllable and observable) realization and must be taken into account. This is taken care of by considering every set of controllability indices in $\mathcal{M}(p)$. Second, while all matrix pairs $(A,Q)_p$ of the form (2.55)-(2.56) are (trivially) completely controllable, the observability of a matrix pair $(A,D)_p$ of the form (2.55) - (2.57) depends on the values of the parameters $\{\gamma_{ijk}\}$ in A and on the columns of D. Therefore, the observability of the realization is investigated in Step 5. ∇

The need for Step 5 in Algorithm (3.36) is evident once it is realized that the situations described in Remark (3.35) indeed occur (especially when the dimension of the desired partial realization is slightly larger than p_M). The following two examples illustrate the application of Algorithm (3.36) and the occurrence of situations (3.35b) and (3.35c).

(3.40) Example. Consider the following partial matrix sequence.

$$L_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad L_2 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad L_3 = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

The minimal realization (obtained using Algorithm (2.78) is unique, and is given by the triple

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

It is desired to construct all partial realizations of dimension p = 3.

The set $M(3)$ consists of $\{2,1\}$ and $\{1,2\}$. The Hankel array for the given finite matrix sequence is

$$\mathcal{H}_{33} = \begin{bmatrix} 1 & 0 & 1 & -1 & 1 & -2 \\ -1 & 1 & -1 & 2 & -1 & 3 \\ 1 & -1 & 1 & -2 & * & * \\ -1 & 2 & -1 & 3 & * & * \\ 1 & -2 & * & * & * & * \\ -1 & 3 & * & * & * & * \end{bmatrix}$$

Consider first the set of controllability indices $\{2,1\}$. The corresponding matrix L is

$$L = [\underline{h}_{11} \quad \underline{h}_{12} \quad \underline{h}_{21}]$$

and the equations for the $\{\gamma_{ijk}\}$ parameters are

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{111} \\ \gamma_{112} \\ \gamma_{121} \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \\ -2 \\ 3 \end{bmatrix} = - \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \gamma_{211} \\ \gamma_{212} \\ \gamma_{221} \end{bmatrix}$$

Notice that there is one degree of freedom in each set of equations. The parameters γ_{112} and γ_{212} can be selected as the arbitrary variables. Then the remaining four parameters are obtained as

$$\gamma_{111} = -1 - \gamma_{112} \qquad \gamma_{211} = 1 - \gamma_{212}$$

$$\gamma_{121} = 0 \qquad \gamma_{221} = -1$$

All partial realizations of dimension equal to three and controllability indices $\{2,1\}$ are represented by the triple

$$A = \begin{bmatrix} 0 & 1+\gamma_{112} & -1+\gamma_{212} \\ 1 & -\gamma_{112} & -\gamma_{212} \\ 0 & 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

where γ_{112} and γ_{212} are arbitrary. The observability matrix (3.39) of this triple is

$$\hat{O} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

which has rank equal to two (by inspection). Thus, all possible realizations with this input structure are unobservable. It is easy to verify that the unobservable pole is at $-1-\gamma_{112}$.

Consider now the set of controllability indices $\{1,2\}$. Matrix L for this case is given by

$$L = [\underline{h}_{11} \quad \underline{h}_{21} \quad \underline{h}_{22}]$$

and the parameters $\{\gamma_{ijk}\}$ are obtained by solving the following equations:

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \gamma_{111} \\ \gamma_{121} \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} = - \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \gamma_{211} \\ \gamma_{221} \\ \gamma_{222} \end{bmatrix}$$

The solution to these equations is given by

$$\gamma_{111} = -1 \qquad \gamma_{211} = 2 + \gamma_{222}$$

$$\gamma_{121} = 0 \qquad \gamma_{221} = -1 - \gamma_{222}$$

with γ_{222} arbitrary. All possible partial realizations of dimension equal to three and controllability indices {1,2} are represented by the triple

$$A = \begin{bmatrix} 1 & 0 & -2-\gamma_{222} \\ 0 & 0 & 1+\gamma_{222} \\ 0 & 1 & -\gamma_{222} \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

with γ_{222} completely arbitrary. The observability matrix (3.39) for the above realization is given by

$$\hat{O} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \\ 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$$

It is readily verified that $\hat{O}$ has rank equal to two; therefore, the realization is unobservable for all values of γ_{222}. The unobservable pole is found to be at $-1-\gamma_{222}$. Thus, all partial realizations of dimension p = 3 are unobservable.

It can be verified that partial realizations of dimension equal to four do exist. In fact, there are completely controllable and observable realizations for every one of the three elements of $M(4)$. All realizations of dimension p = 4 and input structure {2,2} are represented by the triple

$$A = \begin{bmatrix} 0 & 1+\gamma_{112}-\gamma_{122} & 0 & -2+\gamma_{212}-\gamma_{222} \\ 1 & -\gamma_{112} & 0 & -\gamma_{212} \\ 0 & \gamma_{122} & 0 & 1+\gamma_{222} \\ 0 & -\gamma_{122} & 1 & -\gamma_{222} \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 2 \end{bmatrix}$$

where γ_{112}, γ_{122}, γ_{212}, and γ_{222} are arbitrary. It is easy to verify that any such realization is completely observable if and only if $\gamma_{122} \neq 0$. ∇

(3.41) Example. Consider the following partial matrix sequence

$$L_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad L_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Its minimal realization (obtained using Algorithm (2.78)) is unique, has dimension $p_M = 3$, and controllability indices $\{\mu_{1M}, \mu_{2M}\} = \{2,1\}$. It is given by the triple

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

It is desired to construct all partial realizations of dimension $p = 4$.

The set $\mathcal{M}(4)$ contains only two elements: $\{2,2\}$ and $\{3,1\}$. The Hankel array corresponding to the given sequence is

$$\mathcal{H}_{33} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 & * & * \\ 0 & 1 & 0 & 1 & * & * \\ 1 & -1 & 1 & -1 & * & * \\ 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 1 & -1 & * & * & * & * \end{bmatrix}$$

Consider first the set of controllability indices {2,2}. For this choice of controllability indices, matrix L is given by

$$L = [\underline{h}_{11} \quad \underline{h}_{12} \quad \underline{h}_{21} \quad \underline{h}_{22}]$$

and the corresponding equations (3.38) are

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \gamma_{111} \\ \gamma_{112} \\ \gamma_{121} \\ \gamma_{122} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \gamma_{211} \\ \gamma_{212} \\ \gamma_{221} \\ \gamma_{222} \end{bmatrix}$$

There is a total of 2 degrees of freedom in the above sets of equations. Let γ_{122} and γ_{222} be the free parameters. Then,

$$\gamma_{111} = -1 - \gamma_{122} \qquad \gamma_{211} = -\gamma_{222}$$

$$\gamma_{112} = -1 + \gamma_{122} \qquad \gamma_{212} = 1 + \gamma_{222}$$

$$\gamma_{121} = -\gamma_{122} \qquad \gamma_{221} = -1 - \gamma_{222}$$

All (completely controllable) realizations with controllability indices $\{2,2\}$ are represented by the triple

$$A = \begin{bmatrix} 0 & 1+\gamma_{122} & 0 & \gamma_{222} \\ 1 & 1-\gamma_{122} & 0 & -1-\gamma_{222} \\ 0 & \gamma_{122} & 0 & 1+\gamma_{222} \\ 0 & -\gamma_{122} & 1 & -\gamma_{222} \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

where γ_{122} and γ_{222} are arbitrary. The observability matrix (3.39) of this realization is given by (after some simple row operations)

$$\hat{O} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix $\hat{O}$ has full rank, so all the realizations of dimension 4 and controllability indices $\{2,2\}$ are completely controllable and observable. It can further be verified that all of these realizations are unstable.

Consider now the set of controllability indices $\{3,1\}$. Since $\mu_1 = N_0$, the matrix $\hat{M}_1$ is undefined and the vector

$$\underline{\gamma}_1^T = [\gamma_{111} \quad \gamma_{112} \quad \gamma_{113} \quad \gamma_{121}]$$

is arbitrary. Also, $\mu_2 = \mu_{2M}$, so $\underline{\gamma}_2$ is the same as in the minimal realization:

$$\underline{\gamma}_2^T = [\gamma_{211} \quad \gamma_{212} \quad \gamma_{221}] = [-1 \quad 1 \quad -1]$$

Then, all (completely controllable) realizations with controllability indices {3,1} are represented by the triple

$$A = \begin{bmatrix} 0 & 0 & -\gamma_{111} & 1 \\ 1 & 0 & -\gamma_{112} & -1 \\ 0 & 1 & -\gamma_{113} & 0 \\ 0 & 0 & -\gamma_{121} & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Matrix $\hat{\mathcal{O}}$ is constructed next. After a few simple row operations, it is given by

$$\hat{\mathcal{O}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\gamma_{121} & 1 \\ 0 & 0 & -\gamma_{111}-\gamma_{113}-1 & 1 \\ 0 & 0 & -\gamma_{112}-\gamma_{113}-2 & -1 \end{bmatrix}$$

$\hat{\mathcal{O}}$ has full rank if and only if any one of the following three conditions is satisfied

$$\gamma_{121} \neq 0$$

$$\gamma_{111} + \gamma_{113} \neq -1$$

$$\gamma_{112} + \gamma_{113} \neq -2$$

In other words, the values of the elements of the vector $\underline{\gamma}_1$ that do not meet the above requirements produce an unobservable realization. It can be verified that the four arbitrary parameters in this realization are not sufficient to produce completely arbitrary or even stable poles

(that is, all completely controllable and observable realizations with controllability indices {3,1} are unstable). ∇

The algorithm outlined below gives all the minimal partial stable realizations of a given sequence. It consists basically of applying Algorithm (3.36) several times.

(3.42) Minimal Partial Stable Realization Algorithm.

Step 1. Compute the minimal partial realization (realizations, if non-unique) of the given sequence using Algorithm (2.78), and check (by any convenient means) whether it is stable (any are stable, if non-unique). If any realization of dimension p_M is stable, the minimal partial stable realization problem has been solved and there is no need to continue the procedure. Otherwise, proceed to the next step.

Step 2. Find the smallest value of $p \geqq p_M + 1$ for which condition (3.31b) is satisfied.

Step 3. Form the set $\mathcal{M}(p)$ and use Algorithm (3.36) to obtain all partial realizations of dimension p.

Step 4. If partial realizations of dimension p do not exist, increase p by one and repeat Step 3. Otherwise, proceed in a systematic manner to check (by any convenient means) whether the completely controllable and observable realizations of dimension p are stable or not. If any of these realizations is found to be stable, the procedure is finished. Otherwise, increase p by one and repeat Step 3.

Proof. The proof is obvious. ∇

Often, only one minimal stable realization is required. Such may be the case, for example, in the application of partial realization theory to the design of minimal order observers (discussed in the next

chapter). In these cases, the problem is often simplified because one can choose the most convenient set of controllability indices in $M(p)$ for the input structure of the realization. This is illustrated in the following example, which appeared in a paper by Tether (1970).

(3.43) Example. It is desired to obtain a minimal partial stable realization for the following sequence

$$L_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad L_2 = \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix} \quad L_3 = \begin{bmatrix} 10 & 7 \\ 1 & 1 \end{bmatrix}$$

$$L_4 = \begin{bmatrix} 22 & 15 \\ 3 & 3 \end{bmatrix}$$

The Hankel array $\mathcal{H}_{44}$ for this sequence is

$$\mathcal{H}_{44} = \begin{bmatrix} 1 & 1 & 4 & 3 & 10 & 7 & 22 & 15 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 & 3 \\ 4 & 3 & 10 & 7 & 22 & 15 & * & * \\ 0 & 0 & 1 & 1 & 3 & 3 & * & * \\ 10 & 7 & 22 & 15 & * & * & * & * \\ 1 & 1 & 3 & 3 & * & * & * & * \\ 22 & 15 & * & * & * & * & * & * \\ 3 & 3 & * & * & * & * & * & * \end{bmatrix}$$

The minimal partial realization (obtained using Algorithm (2.78)) is not unique, and has dimension $p_M = 5$, with controllability indices $\{\mu_{1M}, \mu_{2M}\} = \{3,2\}$. It is represented in its most general form by the triple

$$A = \begin{bmatrix} 0 & 0 & -\gamma_{111} & 0 & -\gamma_{211} \\ 1 & 0 & -\gamma_{112} & 0 & -\gamma_{212} \\ 0 & 1 & -\gamma_{113} & 0 & -\gamma_{213} \\ 0 & 0 & -\gamma_{121} & 0 & -\gamma_{221} \\ 0 & 0 & -\gamma_{122} & 1 & -\gamma_{222} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 4 & 10 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

with γ_{112}, γ_{121}, γ_{122}, and γ_{222} completely arbitrary, and

$$\gamma_{111} = 8 - 4\gamma_{112} - \gamma_{121} - 3\gamma_{122} \qquad \gamma_{212} = -\gamma_{222}$$

$$\gamma_{113} = -3 \qquad \gamma_{213} = -1$$

$$\gamma_{211} = -2 \qquad \gamma_{221} = 5 + \gamma_{222}$$

It can be shown that the coefficients of the characteristic polynomial of A satisfy the relation

$$\beta_2 + 3\beta_3 + 7\beta_4 + 15\beta_5 = -31$$

and β_1 is completely arbitrary, for any choice of the arbitrary γ_{ijk} parameters of the realization. Since at least one of the $\{\beta_k\}$ must alway be negative, it follows that all minimal partial realizations are unstable. The best that one can do is to fix four stable poles, but the fifth one is always unstable.

This example has been solved by Tether (1970) and by Ackerman (1972) using their respective realization algorithms. The realization obtained by Ackermann (1972) is represented in its most general form by the triple

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a & 0 & b & c & 3-a \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 1 \\ 4 & 3 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

where a, b, and c are arbitrary. This realization has only three arbitrary poles; the remaining two are always unstable. The realization obtained by Tether (1970) is represented in its most general form by the triple

$$A = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ 6 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & d-13 & 4 & -6d+e+27 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where d and e are arbitrary. This realization has only two arbitrary poles, and the remaining three poles are always unstable. Both of these realizations are contained in the one given previously (obtained via Algorithm (2.78)).

Since the minimal realizations are all unstable, a non-minimal stable realization must be constructed. For any $p \geqq 6$, the necessary condition (3.31b) is satisfied trivially because $N_0 = 4$. The set $\mathcal{M}(p)$ contains only two elements: $\{4,2\}$ and $\{3,3\}$. Only one stable realization is de-

sired, so it is worthwhile to investigate which of these two sets is more convenient.

Using (2.80), the number of arbitrary parameters is found to be equal to seven with the set {4,2}, and eight with the set {3,3}. The dimension of the realization is six, so there seems to be a good chance of obtaining a stable (possibly arbitrary) realization with each of these two sets of admissible controllability indices. Even though it yields one less arbitrary parameter, it is preferable to select the set {4,2} because the position that the arbitrary parameters occupy in A is more ameanable for the analytical verification of stability. This concept is expanded on below.

For the set {4,2}, $\underline{\gamma}_1$ is arbitrary because $\mu_1 = N_0$. Then the poles of the first subsystem (the eigenvalues of A_{11}; see Theorem (2.54) and the comments which follow it) are completely arbitrary. Notice that $\mu_2 = \mu_{2M}$ and $\mu_{1M} > \mu_{2M}$ imply $\underline{\gamma}_2$ is the same as for the minimal realization. Thus, the poles of the second subsystem (the eigenvalues of A_{22}) are given by the roots of the polynomial

$$\gamma_{22}(s) = s^2 + \gamma_{222}s + 5 + \gamma_{222}$$

and choosing $\gamma_{222} > 0$ guarantees stability for the second subsystem. Finally, since the two non-zero elements of A_{21} are arbitrary, an attractive and simple solution is to let $\gamma_{121} = \gamma_{122} = 0$, select $\gamma_{222} > 0$, and choose the remaining arbitrary parameters such that the polynomial

$$\gamma_{11}(s) = s^4 + \gamma_{114}s^3 + \gamma_{113}s^2 + \gamma_{112}s + \gamma_{111}$$

matches any desired stable polynomial. It can be shown that when the subsystems are not forced to be decoupled, the realization of input structure {4,2} has arbitrary poles.

It can also be verified that the set {3,3} gives an arbitrary realization, but the simple decoupled structure can never be made stable. For these reasons, the input structure {4,2} will be used.

The minimal stable realization of dimension 6 and controllability indices {4,2} is represented in its most general form by the triple

$$A = \begin{bmatrix} 0 & 0 & 0 & -\gamma_{111} & 0 & -\gamma_{211} \\ 1 & 0 & 0 & -\gamma_{112} & 0 & -\gamma_{212} \\ 0 & 1 & 0 & -\gamma_{113} & 0 & -\gamma_{213} \\ 0 & 0 & 1 & -\gamma_{114} & 0 & 0 \\ 0 & 0 & 0 & -\gamma_{121} & 0 & -\gamma_{221} \\ 0 & 0 & 0 & -\gamma_{122} & 1 & -\gamma_{222} \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 4 & 10 & 22 & 1 & 3 \\ 0 & 0 & 1 & 3 & 0 & 0 \end{bmatrix}$$

with $\gamma_{111}, \ldots, \gamma_{122}$, and γ_{222} completely arbitrary, and

$$\gamma_{211} = -2 \qquad \gamma_{213} = -1$$

$$\gamma_{212} = -\gamma_{222} \qquad \gamma_{221} = 5 + \gamma_{222}$$

A stable realization with poles at -1, -1, -1, -2, $-1 \pm j2.45$ (approximately) is obtained for the following values of the arbitrary parameters

$$\gamma_{111} = 2 \qquad \gamma_{114} = 5$$

$$\gamma_{112} = 7 \qquad \gamma_{121} = \gamma_{122} = 0$$

$$\gamma_{113} = 9 \qquad \gamma_{222} = 2$$

This realization is both completely controllable and observable. ∇

A minimal partial arbitrary realization algorithm is given next. The algorithm follows closely the pattern of Algorithm (3.42), but the stronger results that are available for the arbitrary realization case

often lead to an easier solution in the form of less iterations. Before proceeding to state the algorithm, it is convenient to introduce the following definition.

(3.44) Definition. The subset of $M(p)$ that contains all the sets $\{\mu_i\}$ for which condition (3.19b) is satisfied is denoted by $U(p)$. That is,

$$U(p) = \{\{\mu_i\} : \{\mu_i\} \in M(p) \text{ and } \mu_1 + \mu_2 + \dots + \mu_r \leqq \sigma - \rho(\hat{M}_1) - \rho(\hat{M}_2) - \dots - \rho(\hat{M}_r)\}$$

In words, $U(p)$ is made up of all the admissible sets of controllability indices that yield p or more arbitrary γ_{ijk} realization parameters. In the arbitrary realization context it is useless to consider sets of controllability indices not included in $U(p)$. Notice that the set $U(p)$ need not be a proper subset of $M(p)$; that is, often $U(p) = M(p)$ (as in Example (3.43)). The algorithm follows.

(3.45) Minimal Partial Arbitrary Realization Algorithm.

Step 1. Use Algorithm (2.78) to compute all minimal partial realizations of the given sequence. If unique, proceed to the next step. Otherwise, check whether (3.19b) is satisfied with $p = p_M$ and the minimal controllability indices; if so, check (by any convenient means) whether the minimal partial realization is arbitrary or not. If the minimal realization has arbitrary poles, the minimal partial arbitrary realization problem is solved and there is no need to proceed any further. Otherwise, proceed to the next step.

Step 2. Find the smallest value of $p \geqq p_M + 1$ and the corresponding set $U(p)$ for which condition (3.19b) is satisfied.

Step 3. Use Algorithm (3.36) with $\mathcal{U}(p)$ instead of $\mathcal{M}(p)$ to obtain all partial realizations of dimension p that are possible candidates for minimal partial arbitrary realizations.

Step 4. If partial realizations of dimension p do not exist, increase p by one, form the set $\mathcal{U}(p)$, and repeat Step 3. Otherwise, proceed to check (by any convenient means) in a systematic manner whether the completely controllable and observable realizations of dimension p have arbitrary poles. If any of these realizations has completely arbitrary poles, the procedure is finished. Otherwise, increase p by one, form the set $\mathcal{U}(p)$, and repeat Step 3.

Proof. The proof is obvious. ∇

The algorithms presented in this section are most useful when none of the conditions mentioned in Remark (3.34) occur. Whenever any one of these conditions occurs, it is simpler to use the construction outlined in the proof of Theorem (3.22), or a modified version of Algorithm (3.36). The modifications to be done to Algorithm (3.36) depend on the specific problem at hand, but it is not (generally) difficult to recognize what should be modified, as is the case in the following examples.

(3.46) Example. It is desired to obtain the minimal partial arbitrary realization of the following (row) vector sequence

$$\underline{\zeta}_1 = [1 \quad 0 \quad 1] \qquad \underline{\zeta}_2 = [0 \quad 1 \quad 0] \qquad \underline{\zeta}_3 = [1 \quad 1 \quad -1]$$

$$\underline{\zeta}_4 = [0 \quad 0 \quad 0]$$

From Corollary (3.18), the dimension of the minimal partial arbitrary realization is four, and the realization can be immediately constructed by inspection as the dual of (3.12)-(3.14). It is given by the triple

$$A = \begin{bmatrix} \underline{i}_2^T \\ \underline{i}_3^T \\ \underline{i}_4^T \\ -\underline{\beta}^T \end{bmatrix} \qquad Q = \begin{bmatrix} \underline{\zeta}_1^T \\ \underline{\zeta}_2^T \\ \underline{\zeta}_3^T \\ \underline{\zeta}_4^T \end{bmatrix} \qquad \underline{d}^T = \underline{i}_1^T$$

with $\underline{\beta}^T = [\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4]$ a completely arbitrary vector. ∇

(3.47) Example. It is desired to obtain the minimal partial arbitrary realization of the following matrix sequence

$$L_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \quad L_2 = \begin{bmatrix} 2 & 0 \\ -2 & 0 \\ 0 & 0 \end{bmatrix} \quad L_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The corresponding Hankel array $\mathcal{H}_{33}$ is

$$\mathcal{H}_{33} = \begin{bmatrix} 1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 & * & * \\ -2 & 0 & 0 & 1 & * & * \\ 0 & 0 & 1 & 1 & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 1 & 1 & * & * & * & * \end{bmatrix}$$

Almost immediately it is recognized that $p_M = rN_0 = 6$ and $\mu_{1M} = \mu_{2M} = 3$. Then, from Theorem (3.22) it follows that the minimal partial realization has completely arbitrary poles ($p_A = p_S = p_M$) and can be obtained by inspection. It is given by the triple

$$A = [\underline{i}_2 \quad \underline{i}_3 \quad -\underline{\gamma}_1 \quad \underline{i}_5 \quad \underline{i}_6 \quad -\underline{\gamma}_2] \qquad Q = [\underline{i}_1 \quad \underline{i}_4]$$

$$D = [\underline{\ell}_{11} \quad \underline{\ell}_{12} \quad \underline{\ell}_{13} \quad \underline{\ell}_{21} \quad \underline{\ell}_{22} \quad \underline{\ell}_{23}]$$

where

$$\underline{\gamma}_1^T = [\gamma_{111} \quad \gamma_{112} \quad \gamma_{113} \quad \gamma_{121} \quad \gamma_{122} \quad \gamma_{123}]$$

$$\underline{\gamma}_2^T = [\gamma_{211} \quad \gamma_{212} \quad \gamma_{213} \quad \gamma_{221} \quad \gamma_{222} \quad \gamma_{223}]$$

are completely arbitrary vectors. ∇

It is evident in Example (3.46) that the observability index, δ, and the vectors $\{\underline{\zeta}_i^T\}$ are invariants of the minimal partial arbitrary realization of the given (row) vector sequence. Notice also that the controllability indices $\{\mu_i\}$ and the vectors $\{\underline{\ell}_{jk}\}$ are invariants of the minimal partial, the minimal partial stable, and the minimal partial arbitrary realization of the matrix sequence of Example (3.47). If the partial realization with controllability indices $\{3,3\}$ is constructed for the matrix sequence of Example (3.43), it is found that the vectors $\underline{\ell}_{11}$, $\underline{\ell}_{12}$, $\underline{\ell}_{13}$, $\underline{\ell}_{21}$, and $\underline{\ell}_{22}$ are minimal partial stable and minimal partial arbitrary realization invariants, while the vectors $\underline{\ell}_{14}$ and $\underline{\ell}_{23}$ and the controllability indices are not. From these observations and the results of Chapter 2 on minimal partial realizations, it is apparent that whether or not the parameters $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ and the vectors $\{\underline{\ell}_{jk}\}$ are minimal partial, minimal partial stable and/or minimal partial arbitrary realization invariants is dependent exclusively upon the given matrix sequence. These concepts are made more precise in the following remark.

(3.48) Remark. Recall that a complete system of invariants (for the equivalence relation defined by a change of basis on the state space) for an infinite matrix sequence that admits a finite-dimensional realization consists of the parameters $\{\mu_i\}$ and $\{\gamma_{ijk}\}$ and the vectors $\{\underline{\ell}_{jk}\}$. It would be very convenient if analogous results could be stated for a finite matrix sequence and its corresponding minimal partial, minimal partial stable, and minimal partial arbitrary realizations, but this

does not seem to be the case. Nevertheless, some invariants (for more general equivalence relations) can be identified for these problems.

Let $\mathcal{S}_M(p_M)$ be the set of all completely controllable and observable matrix triples $(A,Q,D)_{p_M}$ that satisfy condition (2.5a) for a given matrix sequence of N_0 elements, $\mathcal{S}_S(p_S)$ be the set of all completely controllable and observable matrix triples $(A,Q,D)_{p_S}$ that satisfy conditions (3.1a) and (3.1c) for a given matrix sequence of N_0 elements, and $\mathcal{S}_A(p_A)$ be the set of all completely controllable and observable matrix triples $(A,Q,D)_{p_A}$ that satisfy conditions (3.2a) and (3.2c) for a given matrix sequence of N_0 elements. Then the following statements can be made.

a) The parameters $\{\mu_{iM}\}$, the vectors $\{\underline{\ell}_{jk}\}$, and those parameters $\{\gamma_{ijk}\}$ that have fixed numerical values are invariants for an appropriately-defined equivalence relation on $\mathcal{S}_M(p_M)$. These invariants form a complete system of invariants for the equivalence relation defined by GL(p) acting on $\mathcal{S}_M(p_M)$ if and only if the minimal partial realization of the sequence is unique (in other words, when all the $\{\gamma_{ijk}\}$ parameters have fixed numerical values).

b) The vectors $\{\underline{\ell}_{jk}\}$ for $j = 1,2,\ldots,r$ and $k = 1,2,\ldots,\mu_{jM}$ only, and those parameters $\{\gamma_{ijk}\}$ that have fixed numerical values are invariants for an appropriately-defined equivalence relation on $\mathcal{S}_S(p_S)$ when $p_S > p_M$. In some cases it is possible that one or more parameters $\{\mu_i\}$ and vectors $\{\underline{\ell}_{jk}\}$ for some $1 \leqq j \leqq r$ and $k = \mu_{jM},\mu_{jM+1},\ldots,\mu_j$ are also invariants. These invariants form a complete system of invariants for the

equivalence relation defined by GL(p) acting on $\mathcal{S}_M(p_M)$ if and only if $p_S = p_M$(which implies $\mathcal{S}_S(p_S) = \mathcal{S}_M(p_M)$) and the minimal partial realization is unique.

c) The vectors $\{\underline{\ell}_{jk}\}$ for j = 1,2,...,r and k = 1,2,..., μ_{jM} only, and those parameters $\{\gamma_{ijk}\}$ that have fixed numerical values are invariants for an appropriately-defined equivalence relation on $\mathcal{S}_A(p_A)$ when $p_A > p_M$. In some cases it is possible that one or more parameters $\{\mu_i\}$ and vectors $\{\underline{\ell}_{jk}\}$ for some $1 \leqq j \leqq r$ and $k = \mu_{jM}, \mu_{jM+1}, \ldots, \mu_j$ are also invariants.

The form of the equivalence relation that applies in the above statements is probably not easily specified, in general. More research in this area is needed. ∇

The realization theory results presented thus far have a significant application in the design of minimal order observers discussed in the following chapter. As a matter of fact, it was the observer design problem which promoted the formulation and present discussion of the minimal partial stable and the minimal partial arbitrary realization problems. As mentioned above, the treatment given here to these problems is by no means final. Several interesting questions remain open, and it is likely that even more questions will be uncovered in the future. It also seems that the realization theory results of these two chapters (2 and 3) will find application in other hitherto unrelated areas (besides observer design). These comments are expanded in Chapter 5.

A discussion of the numerical aspect of the solution to these problems is not within the scope of this work, although the practical importance of such considerations is recognized. Observe that there are

no conceptual difficulties involved in the application of algorithms (3.36), (3.42), and (3.45). However, it is not a simple task to verify observability for a triple in the form (2.55)-(2.57), and to check whether matrix A has stable or arbitrary poles when several of the elements of the pair $(A,D)_p$ are unspecified or are expressed as a function (possibly nonlinear) of other unspecified elements. Success or failure in solving such problems depends on the experience and ingenuity of the person solving the problem, and on the numerical technique used.

CHAPTER 4
DESIGN OF OBSERVERS VIA REALIZATION THEORY

In this chapter the design of low order observers to estimate vector linear functions of the state of a multivariable linear system is discussed. The problem is defined formally and a solution is proposed from a completely new point of view. Three special problems are considered in detail: the design of minimal order (stable) observers, the design of minimal order observers with arbitrary poles, and the design of intermediate order observers. It is shown that the structure and dynamics of these observers depend exclusively on the output structure of the observed system and on the given feedback control law. The design procedure presented here gives all the possible observers of specified dimension for a given system and control law, and the observer system matrix is obtained in the form (2.55).

Loosely speaking, an observer is a system which processes (on-line) the information available in a system's inputs and outputs to provide an asymptotic approximation (in the sense that the error goes to zero as time increases) of a vector linear function of the given system's state. This makes observers be of significant importance because the solution to most modern control algorithms (for example, linear quadratic loss, pole placement, decoupling, etc.) is generally expressed as a linear feedback function of the state, and only seldom is the complete state available for feedback. In the cases that the complete state is not available, the output of an observer together with the observed system output can be used as a sufficiently accurate replacement. The

dimension of an observer which estimates the complete state is equal to (or larger than) the number of system states minus the number of its linearly independent outputs, and the poles of the observer can be chosen practically at will (the only requirement is that of stability, although certain other considerations may be important; see Bongiorno and Youla, 1968, 1970, and Bongiorno, 1973). If the feedback control function to be implemented is known precisely and beforehand, it is usually possible to design an observer of smaller dimension at the expense of some loss in arbitrariness of the observer pole positions. In most cases, the reduction in the order of the observer more than justifies the loss in arbitrariness of its pole positions.

The observer design problem was formulated and solved first in 1964 by Luenberger (1964), who proposed a design procedure for single output systems only. Two years later, Luenberger (1966) extended his original results to systems with several outputs, and considered the problem of observing a scalar linear function of the state. Since then, a significant number of new results and extensions have been discussed in the literature (see Luenberger, 1971, for a state-of-the-art tutorial and survey of results; see also Chapter 1 of this work for a discussion of post-1971 developments). As of today, the problem of designing minimal order observers to estimate the complete state has received most of the attention, and there are several techniques available to carry out this design. Some of the most significant ones are those of Bongiorno and Youla (1968), Johnson (1969), Retallack (1970), Gopinath (1971), Jameson and Rothschild (1971), and Wolovich (1973a). Results in the design of minimal (or reduced) order observers to estimate vector linear functions of the state have not been so prolific. Most of the results for this

problem have been obtained very recently (1970-1974), and are, at best, only partial. The most relevant work for this specific problem has been done by Haley (1967), Williamson (1970), Rothschild and Grammaticos (1971), Fortmann and Williamson (1972), Wang and Davison (1973) and Roman and Bullock (1974).

The contributions of this chapter to observer theory are manifold. It is shown that the solution to an observer design problem can be obtained (in all cases) via the solution of an appropriately formulated partial realization problem. This close relationship between observer theory and realization theory is reported here for the first time. A solution is given for the special cases where a vector linear function of the state is to be estimated with an observer of: a) minimal order and stable poles, b) minimal order and arbitrary poles, and c) specified intermediate dimension and stable poles. These problems have been discussed to some extent in the literature (see the references cited in the preceding paragraph), but the currently available results are far from complete. The discussion given here is thorough, and the proposed design procedures can always be used to obtain the desired answer in all three of these cases. Finally, several of the previous observer design techniques are examined in the realization theory context, and it is shown that each corresponds to a specific type of partial realization algorithm. Thus, the realization theory setting can be used as a means for comparing and evaluating different observer design techniques.

Problem Statement

In this section the observer design problem is formulated and the basic results in observer theory are presented. These results are not

proved or discussed extensively, since they are well known and readily available in the literature.

The systems considered are defined by equations of the form (without loss of generality)

(4.1a) $$\dot{\underline{x}} = F\underline{x} + G\underline{u} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \underline{x} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \underline{u}$$

(4.1b) $$\underline{y} = H\underline{x} = [0_{r,n-r} \quad I_r]\underline{x}$$

where $\underline{x}$ is the n-dimensional state vector, $\underline{u}$ is the m-dimensional input vector, $\underline{y}$ is the r-dimensional output vector, and F, G, H are matrices of appropriate dimensions with partitions defined as follows: F_{11} is $(n-r)\times(n-r)$, F_{12} is $(n-r)\times r$, F_{21} is $r\times(n-r)$, F_{22} is $r\times r$, G_1 is $(n-r)\times m$, and G_2 is $r\times m$. It is assumed that (4.1) represents a completely observable system, and that $r < n$. System (4.1) will also be denoted by the triple $(F,G,H)_n$, and occasionally referred to as the plant.

Suppose now that a feedback control law of the form

(4.2) $$\underline{u} = K\underline{x} = [K_1 \quad K_2]\underline{x}$$

where K is an m×n matrix, K_1 is $m\times(n-r)$, and K_2 is $m\times r$, is specified for implementation. More precisely, the following problem is to be solved.

(4.3) Definition. Given a completely observable plant of the form (4.1) and a feedback control law of the form (4.2), find a method to implement the given control law so that the overall closed loop system behaves as if complete state variable feedback is used:

(4.4a) $$\dot{\underline{x}} = [F + GK]\underline{x} + G\underline{v}$$

(4.4b) $$\underline{y} = H\underline{x}$$

where $\underline{v}$ is an m-dimensional external input.

This problem (expressed in a different form) is probably the oldest problem in control theory. It corresponds to the classical compensator transfer function design problem, and has been treated extensively in the classical control literature for single input/output systems (see, for example, D'Azzo and Houpis, 1966; Saucedo and Schiring, 1968). The classical control methods (Root locus, Bode plots, Nyquist and Nichols diagram, etc.) are adequate for single input/output systems, but become close to impossible to work with for multivariable systems. This is due to the fact that most classical techniques are analysis tools rather than design tools.

The advent of state variable methods in the late '50s and early '60s shed new light into the solution of problem (4.3) and similar problems because it became possible to formulate these questions in a more precise and mathematical manner, and consequently, a wealth of results in several mathematical fields (linear algebra, calculus of variations, functional analysis, etc.) were readily adopted.

In the context of modern control theory, several solutions have been proposed for problem (4.3). Notably are those of Pearson (1969), Pearson and Ding (1969), Ferguson and Rekasius (1969), Brasch and Pearson (1970), Chen (1970), and Morse and Wonham (1970). However, none of these solutions seem to enjoy the advantages of the so-called observers (the treatment of initial conditions, the straightforward assignment of the observer poles, and other significant properties). Moore (1970) gives an interesting comparison of observers and several alternate compensator design techniques. The material discussed in this work is limited to observers only. The results given in the first three sections of this chapter are the subject of a forthcoming paper (Roman and Bullock, 1975).

Consider a system of the form

(4.5a) $\qquad \dot{\underline{z}} = A\underline{z} + B\underline{u} + C\underline{y}$

(4.5b) $\qquad \underline{w} = D\underline{z} + E\underline{y}$

where $\underline{z}$ is the p-dimensional state vector, $\underline{w}$ is the m-dimensional output vector, A, B, C, D, E are matrices of appropriate dimensions, and $\underline{u}$, $\underline{y}$ are as previously defined. System (4.5) is also represented by the quintuple $(A,B,C,D,E)_p$.

(4.6) Definition. System (4.5) is said to be an observer for the feedback control law (4.2) of system (4.1) if the output of (4.5) provides an asymptotic estimate of $K\underline{x}$ in the sense that

(4.7) $$\underline{w} = K\underline{x} + \underline{\varepsilon}_1$$

where $\underline{\varepsilon}_1 \to \underline{0}$ as $t \to \infty$.

It seems reasonable to expect that if (4.7) is true, then using $\underline{w}$ instead of $K\underline{x}$ yields a satisfactory closed loop system, at least in the steady-state. This indeed turns out to be the case, and satisfactory results are obtained even under transient conditions (Luenberger, 1964; also Anderson and Moore, 1971). Notice that it is pointless to design an uncontrollable and/or unobservable observer, because the corresponding reduced order system (after removing uncontrollable and/or unobservable states) provides the same estimate of $K\underline{x}$.

The following theorem contains the fundamental results of observer theory. The proof is omitted since it is available (in parts) in the literature (see Luenberger, 1964, 1966, 1971, or Fortmann and Williamson, 1972).

(4.8) Theorem. System (4.5) is an observer for the feedback control law (4.2) of system (4.1) if and only if $(F,H)_n$ is a

completely observable pair and the following conditions are satisfied:

a) $B = TG$

b) $CH = TF - AT$

c) $K = DT + EH$

d) all the eigenvalues of A have negative real parts

where T is a $p \times n$ matrix that relates the state of (4.1) to the state of (4.5) as follows

(4.9) $$\underline{z} = T\underline{x} + e^{At}[\underline{z}(0) - T\underline{x}(0)] = T\underline{x} + \underline{\varepsilon}$$

$\underline{z}(0)$ and $\underline{x}(0)$ are the initial conditions on (4.5) and (4.1), respectively, and $\underline{\varepsilon} \to \underline{0}$ as $t \to \infty$ (in view of (4.8d)). Also, $\underline{\varepsilon}_1 = D\underline{\varepsilon}$. Further, T is unique if and only if A and F have no eigenvalues in common, and the $n + p$ poles of the closed loop system obtained using $\underline{u} = \underline{w}$ in (4.1) and (4.5) are given by the eigenvalues of $F + GK$ and A.

Theorem (4.8) also includes the case where the complete state (rather than $K\underline{x}$) must be estimated. This special case is considered by replacing K in (4.8c) with the n-dimensional identity matrix. In this sense, the above theorem is completely general.

The following remark is important.

(4.10) Remark. Observe that conditions (4.8a)-(4.8c) are purely algebraic; the solution of these equations for the quintuple $(A,B,C,D,E)_p$ and matrix T requires mathematical knowledge but very little "guesswork". Notice also that (4.8a) is very easily handled once matrices A, C, D, E, and T are found which satisfy (4.8b)-(4.8d). For this reason, condition (4.8a) can be set aside until T is available. ∇

Theorem (4.8) states the (necessary and sufficient) conditions that must be satisfied by a system of dimension p to be an observer for Kx. That is, given a (completely observable) system (4.1), a feedback control law (4.2), and a system (4.5), then using conditions (4.8a)-(4.8d) one can determine whether or not (4.5) is an observer for the feedback control law (4.2) of system (4.1). But there are still other questions of interest. Several different problems can be posed by placing restrictions on the value of p, or on the allowable observer poles, specifying a desired observer characteristic, or combinations of these. The following two definitions state the most significant specific problems that are discussed in this work.

(4.11) Definition. System (4.5) is a minimal order observer for the feedback control law (4.2) of system (4.1) if the conditions (4.8a)-(4.8d) are satisified, and p is as small as possible.

(4.12) Definition. System (4.5) is a minimal order arbitrary observer for the feedback control law (4.2) of system (4.1) if the conditions (4.8a)-(4.8d) are satisfied, the eigenvalues of A can assume any set of p (stable) values, and p is as small as possible.

It is shown in the sequel that the solution to these and other similar problems can always be obtained through the solution to related minimal partial realization problems, and that the value of p depends on K_1, the output structure of (4.1), and the desired observer characteristics. In order to do so, it convenient to introduce a specific set of coordinates for the state-space of (4.1) and carry out some simplifications on (4.8a)-(4.8c). This is done in the next section.

Preliminary Results

The constraints (4.8a)-(4.8d) that must be satisfied by a quintuple $(A,B,C,D,E)_p$ which defines an observer for $K\underline{x}$ are not as formidable as they appear at first glance. If p is chosen sufficiently large ($p \geq n - r$), then it is not too difficult to solve for the observer matrices (see, for example, Gopinath, 1971; or Luenberger, 1971). Observers of dimension equal to n - r are commonly referred to as full-state observers.

The observers of interest here are minimal order observers, minimal order arbitrary observers, and intermediate order observers (of dimension larger than minimal but smaller than n - r). Therefore, the objective is to solve (4.8b) and (4.8c) subject to A having stable or arbitrary eigenvalues and p being minimal or equal to a specified value (recall that (4.8a) does not place any restrictions on T and consequently, is easily handled once T is obtained). It is shown below that conditions (4.8b) and (4.8c) reduce to a single condition involving only A, D, K_1, and a submatrix of T (defined below). This result is arrived at through the selection of special coordinates for system (4.1). But first it is convenient to carry out some manipulations on (4.8b) and (4.8c).

Define a partition on T as follows

(4.13) $$T = [T_1 \quad T_2]$$

where T_1 is $p\times(n-r)$ and T_2 is $p\times r$. Now substitute the partitioned matrices F, H, K, and T into (4.8b) and (4.8c) to obtain

(4.14) $$C = T_1F_{12} + T_2F_{22} - AT_2$$

(4.15) $$E = K_2 - DT_2$$

(4.16) $$T_2F_{21} = AT_1 - T_1F_{11}$$

(4.17) $$K_1 = DT_1$$

These equations represent a simpler set of equations than (4.8b) and (4.8c) in the following sense. Observe that C and E are defined explicitly in terms of the other matrices. Thus, once matrices A, D, and T are found which satisfy (4.16), (4.17), and the specified constraints on the value of p and eigenvalues of A, then C and E are easily obtained using (4.14) and (4.15), respectively.

The following simple but helpful lemma can be immediately derived from (4.17).

(4.18) Lemma. The dimension of an observer which estimates $K\underline{x}$ for system (4.1) is bounded below by the rank of K_1. That is,

(4.19) $$p \geqq \rho(K_1)$$

The simple proof of Lemma (4.18) is omitted. Bound (4.19) is not too strong in general, but is easy to obtain (it is strongest in the cases where m is very close to n - r). A stronger bound for the dimension of an observer for $K\underline{x}$ is given in Remark (4.43).

Conditions (4.8d), (4.16), and (4.17) determine the allowable A, D, and T matrices for the desired observer properties (minimal order, minimal order and arbitrary poles, intermediate order, etc.). Equations (4.16) and (4.17) are nonlinear equations in the elements of the matrices A, D, and T. Generally, nonlinear equations do not lend themselves to a straightforward analytic solution. In the case at hand the situtation is more complex than usual due to the other requirements imposed on the elements of A. Fortunately, these particular equations can be further simplified considerably.

Notice that (4.16) is of the same form as (4.8b). This similarity is very convenient because equations of the form (4.8b) have been studied extensively in matrix theory by Gantmacher (1969, Vol. 1) and Luenberger (1965), and also specifically in the observer context by Luenberger

(1964, 1966, 1971), Bongiorno and Youla (1968), and Gopinath (1971), among others. So, it is reasonable to expect that one or more of the available techniques can be used to simplify equation (4.16). This indeed turns out to be the case. The discussions and results that follow are basic toward such a development.

(4.20) Remark. The partitions defined on F and H in (4.1) implicitly define a partition on $\underline{x}$ as $\underline{x}^T = [\underline{x}_1^T \quad \underline{x}_2^T]$ where $\underline{x}_1$ is $(n-r)\times 1$ and $\underline{x}_2$ is $r\times 1$. That is, (4.1) is equivalent to

(4.21a) $$\dot{\underline{x}}_1 = F_{11}\underline{x}_1 + F_{12}\underline{x}_2 + G_1\underline{u}$$

(4.21b) $$\dot{\underline{x}}_2 = F_{21}\underline{x}_1 + F_{22}\underline{x}_2 + G_2\underline{u}$$

(4.21c) $$\underline{y} = \underline{x}_2$$

System (4.1) is completely observable, so system (4.21) is completely observable; therefore, $\underline{x}_1$ must be observed from $\underline{x}_2$. Intuitively, the pair $(F_{11},F_{21})_{n-r}$ must be a completely observable pair. This is indeed true, as stated in the following lemma due to Gopinath (1971). ∇

(4.22) Lemma. Consider a linear system with partitions defined as in (4.1). If the pair $(F,H)_n$ is completely observable, then the pair $(F_{11},F_{21})_{n-r}$ is also completely observable.

The proof of Lemma (4.22) consists of performing several simple row operations on the observability matrix of (4.1). The details are omitted here for brevity.

A simple but important corollary is obtained as a by-product of the proof of Lemma (4.22). It is given after the following definition.

(4.23) Definition. The set of observability indices of system (4.1) is denoted by the set $\{\nu_i\}$; correspondingly, $\nu = \max(\nu_i)$ is the observability index of (4.1). On occassions, the integer $\nu - 1$ is denoted as N_0.

(4.24) Corollary. The set of observability indices of the matrix pair $(F_{11}, F_{21})_{n-r}$ is given by the set $\{\upsilon_i : \upsilon_i = \nu_i - 1,\ i = 1,2,\ldots,r\}$.

Corollary (4.24) allows the application of a well-known result to the matrix pair $(F_{11}, F_{21})_{n-r}$. This is the following lemma.

(4.25) Lemma. If the matrix pair $(F_{11}, F_{21})_{n-r}$ is a completely observable pair, then it can be represented in a specific basis as follows

(4.26)
$$F_{11} = \left[\begin{array}{c:c:c:c:c}
\underline{0}^T_{\upsilon_1 - 1} & & & \underline{0}^T_{\upsilon_r - 1} & \\
\hdashline
I_{\upsilon_1 - 1} & & & \vdots & \\
\hdashline
\underline{0}^T_{\upsilon_1 - 1} & \underline{f}_1 & \cdots & \hdashline \underline{0}^T_{\upsilon_r - 1} & \underline{f}_r \\
\hdashline
\vdots & & & \underline{0}^T_{\upsilon_r - 1} & \\
\hdashline
\underline{0}^T_{\upsilon_1 - 1} & & & I_{\upsilon_r - 1} &
\end{array}\right]$$

(4.27)
$$F_{21} = \begin{bmatrix} \underline{i}^T_{\upsilon_1} \\ \underline{i}^T_{\upsilon_1 + \upsilon_2} \\ \vdots \\ \underline{i}^T_{n-r} \end{bmatrix}$$

where $\underline{f}_i$ is an (n-r)-dimensional vector.

The proof of this lemma is also omitted because it does not contribute to the present discussion and there are several proofs available in the literature (see Luenberger, 1967, or Roman et al., 1973, for two different proofs).

(4.28) Remark. In (4.26) and (4.27) it has been implicitly assumed that $\upsilon_i \geq 1$ for $i = 1,2,\ldots,r$. This assumption is introduced only for notational simplicity, and is used throughout the rest of this dissertation. It is also assumed in the sequel (without loss of generality) that the pair $(F_{11},F_{21})_{n-r}$ is given in the form (4.26)-(4.27). K, G, F_{12}, and F_{22} are also assumed to be given in the corresponding coordinates. ∇

(4.29) Remark. It is important to note that the form given in Lemma (4.25) is different from the form described in Theorem (2.54). The form (4.26)-(4.27) for a matrix pair is not canonical (under the action of GL(p)) except in very special cases; namely, when $\mathcal{S}(p)$ is the set of all completely controllable matrix pairs for which $\upsilon_1 = \upsilon_2 \ldots = \upsilon_r$ (see Kalman, 1971a). It is interesting to note that in the particular cases when (4.26)-(4.27) is canonical, the parameters $\{\theta_{ijk}\}$ of the dual of (2.55)-(2.56) and the elements of the vectors $\{\underline{f}_i\}$ are related to each other through a set of simple formulas (see Munro, 1973, 1974).

Luo and Bullock (1975) have shown that some of the elements of the vectors $\{\underline{f}_i\}$ in (4.26) can be set to zero once the $\{\upsilon_i\}$ are known, but this fact is not very significant in the present discussion. ∇

With the aid of Lemma (4.25), it is possible to simplify (4.16) considerably and obtain as a result the columns of T_2 expressed in terms of F_{11}, A, and r specific columns of T_1, and obtain also a set of simple recurrence relations for the columns of T_1.

Let the columns of T_1 and T_2 be labeled as follows

(4.30) $$T_1 = [\underline{t}_{11} \quad \underline{t}_{12} \quad \cdots \quad \underline{t}_{1\upsilon_1} \quad \underline{t}_{21} \quad \cdots \quad \underline{t}_{2\upsilon_2} \quad \cdots \quad \underline{t}_{r1} \quad \cdots \quad \underline{t}_{r\upsilon_r}]$$

(4.31) $$T_2 = [\underline{t}_1 \quad \underline{t}_2 \quad \cdots \quad \underline{t}_r]$$

Then the following result can be stated.

(4.32) Lemma. Consider equation (4.16) and suppose the matrix pair $(F_{11}, F_{21})_{n-r}$ is given in the form (4.16)-(4.27). Then the columns of matrices T_1 and T_2 satisfy the following relations

$$\text{a)} \quad \underline{t}_{ij} = A^{j-1}\underline{t}_{i1} \qquad \begin{aligned} i &= 1,2,\ldots,r \\ j &= 1,2,\ldots,\upsilon_i \end{aligned}$$

$$\text{b)} \quad \underline{t}_i = A^{\upsilon_i}\underline{t}_{i1} + T_1\underline{f}_i \qquad i = 1,2,\ldots,r$$

Proof. Follows by direct substitution of (4.26), (4.27), (4.30), and (4.31) into (4.16). ∇

The relations given in Lemma (4.32) for the columns of T_1 and T_2 are similar to those derived by Luenberger (1966) for the columns of T and C, respectively, in the solution of (4.8b) for the design of a full-state observer. Williamson (1970) and Fortmann and Williamson (1972) have also used similar relations for the design of intermediate order observers.

The following notation is introduced to simplify future explanations and to be able to relate the results of Chapters 2 and 3 to this discussion.

(4.33) Notation. The vectors $\underline{t}_{11}, \underline{t}_{21}, \ldots, \underline{t}_{r1}$ are also denoted by the vectors $\underline{q}_1, \underline{q}_2, \ldots, \underline{q}_r$ which are the columns of Q, a p×r matrix. That is,

$$Q = [\underline{q}_1 \quad \underline{q}_2 \quad \cdots \quad \underline{q}_r] = [\underline{t}_{11} \quad \underline{t}_{21} \quad \cdots \quad \underline{t}_{r1}] \qquad \nabla$$

Using this notation, the results of this section can be summarized as follows: the problem of designing a minimal order (minimal order arbitrary, or intermediate order) observer for the feedback control law (4.2) of system (4.1) reduces (for all practical purposes) to that of

finding matrices A, Q, and D that satisfy (4.17) (in view of (4.32a) and (4.33)), subject to A having stable (or arbitrary) eigenvalues and p being minimal (or intermediate). It is shown in the next section that this statement is equivalently expressed by Definition (2.5) (Definition (3.1), or Definition (3.2)), slightly modified.

Observer Design as a Partial Realization Problem

It is shown below that the developments of the preceding section lead naturally to a reformulation of an observer design problem as a partial realization problem. This renders the results of the preceding chapters (and of realization theory in general) applicable to a previously unrelated problem. It is also shown that as a consequence of the possible disparity in values of the observability indices of $(F,H)_n$, a finite matrix sequence which arises from an observer design problem generally has one or more matrix elements with one or more unspecified columns. At first glance, it may seem that the presence of unspecified columns in the matrices of the sequence create complications when obtaining a realization of the sequence, but it is shown in the next section that this is not the case.

The following theorem is the most significant single result of this dissertation.

(4.34) Theorem. System (4.5) is an observer for the feedback control law (4.2) of system (4.1) if and only if $(F,H)_n$ is a completely observable pair, matrices T, B, C, and E are calculated using (4.32), (4.8a), (4.14), and (4.15), respectively, and matrices A, Q, and D satisfy the following conditions

a) $L_i = DA^{i-1}Q$ $\qquad i = 1,2,\ldots,N_0$

b) all the eigenvalues of A have negative real parts.

Matrix L_i is dimensioned $m \times r$ and has its jth column equal to the $(\upsilon_1+\upsilon_2+\ldots+\upsilon_{j-1}+i)$th column of K_1 for $\upsilon_j \geq i$, and unspecified for $\upsilon_j < i$.

Proof. It suffices to shown that (4.34a) is equivalent to (4.17); the rest follows from Theorem (4.8) and the developments of the preceding section.

Label the columns of K_1 as follows

(4.35) $$K_1 = [\underline{k}_{11} \quad \underline{k}_{12} \quad \cdots \quad \underline{k}_{1\upsilon_1} \quad \underline{k}_{21} \quad \cdots \quad \underline{k}_{2\upsilon_2} \quad \cdots \quad \underline{k}_{r1} \quad \cdots \quad \underline{k}_{r\upsilon_r}]$$

From Definition (4.23) and Corollary (4.24) it follows that $N_0 = \upsilon = \max(\upsilon_i)$. Now define a set of vectors $\{\underline{\ell}_{ji}\}$ for $j = 1,2,\ldots,r$ and $i = 1, 1,\ldots,N_0$ as

(4.36) $$\underline{\ell}_{ji} = \begin{cases} \underline{k}_{ji} & \text{if } \upsilon_j \geq i \\ * & \text{if } \upsilon_j < i \end{cases}$$

where, as before, an asterisk represents an unspecified element. Then, in view of (4.17), (4.28), (4.33), and (4.35) it is true that these vectors satisfy the relations given next:

(4.37) $$\underline{\ell}_{ji} = DA^{i-1}\underline{t}_{j1} = DA^{i-1}\underline{q}_j \qquad \begin{array}{l} j = 1,2,\ldots,r \\ i = 1,2,\ldots,\upsilon_j \end{array}$$

Also since $\underline{\ell}_{ji}$ is unspecified for $i > \upsilon_j$, then the left-hand side of (4.37) for $\upsilon_j + 1 \leq i \leq \upsilon = N_0$ can be set equal to whatever value matches a given right-hand side. Finally, (4.34a) follows by defining $\underline{\ell}_{ji}$ to be the jth column of L_i. ∇

Consider now the following corollary to Theorem (4.34).

(4.38) Corollary. The allowable sets of (stable) poles for an observer $(A,B,C,D,E)_p$ for the feedback control law (4.2) of

system (4.1) are determined exclusively by the output structure of system (4.1) and matrix K_1.

Proof. Follows directly from (4.34a). ∇

(4.39) Remark. The form (4.26)-(4.27) adopted for the pair $(F_{11},F_{12})_{n-r}$ can be viewed as a partitioning of (4.21a) into r subsystems coupled to each other in both directions, with the ith subsystem completely observable from the ith system output.

The partition defined on K in (4.1) assigns the gain matrix K_1 to the states $\underline{x}_1$ and the gain matrix K_2 to the states $\underline{x}_2$ (available directly at the output). It follows that if $K_1 = 0$ an observer is not necessary. A finite matrix sequence constructed from an observer design problem where $K_1 = 0$ is a null sequence, and such sequences are generally meaningless (as argued in Chapter 2).

In the sequel, the vectors $\underline{k}_{j1}, \underline{k}_{j2}, \ldots, \underline{k}_{j\upsilon_j}$ in (4.35) are referred to as the jth chain of matrix K_1. It is important to notice that a chain of K_1 composed of null vectors implies that the states in the corresponding subsystem of (4.21a) are not required for feedback. It is shown subsequently that the occurrence of a null chain in K_1 may lead to a reduction in the dimension of the required observer. ∇

Before proceeding any further it is convenient to adopt the following notational conventions.

(4.40) Notation. In the preceding two chapters, the letter k represented an integer subscript and was associated with the columns of the elements of a matrix sequence (among other things). But since K represents a feedback matrix in this chapter, the letter i will be used in place of k as a subscript in discussions where matrix K (or its elements) are directly involved (as in Theorem (4.34) and its proof).

As before, an asterisk (*) represents an unspecified element (whether it is a matrix or a column of a matrix). Occasionally though, it may be necessary to refer to a certain unspecified element. In such situations, a vector $\underline{\xi}_{ji}$ or a scalar ξ_i is used, and the intended meaning should be clear from the context. ∇

Theorem (4.34) is significant because it reduces an observer design problem to a single equation, the solution of which can be obtained through (partial) realization theory, an area that has been extensively studied in control theory and other fields (see Chapter 1). From another point of view, it provides an application for realization theory and suggests several possibilities for future research in this area, because a realization problem that appears meaningless or useless in the realization context can have significant implications in the observer design context.

Corollary (4.38) also has important implications. Since the actual values of the elements of F, G, and H do not enter into the equation that must be solved for matrix A, it is then possible to study the sets of p stable poles that can observe a given feedback control law for a class of systems that have the same output structure, and other related problems. An example is given at the end of this chapter to illustrate these concepts.

Generally, a partial matrix sequence which is formed with the columns of K_1 has one or more of its matrix elements with one or more unspecified columns. As a matter of fact, it follows from (4.36) that this will occur unless $\nu_1 = \nu_2 = \ldots = \nu_r = \nu = N_0$ (which corresponds to the case where $n = r\nu$). A partial matrix sequence that has one or more of its matrix elements with some unspecified columns is referred to as an incompletely specified finite (or partial) matrix sequence.

As a result of Theorem (4.34), the partial realization theory discussions and results of the two preceding chapters can be used to design a low order (minimal, minimal with arbitrary poles, or intermediate) observer for a given feedback control law. The realization theory result corresponding to a specific observer design problem is obvious (for example, a minimal order observer is designed using Algorithm (3.42), the minimal partial stable realization algorithm); however, several of the comments and results in Chapters 2 and 3 need to be interpreted differently in the observer design problem context, and some considerations have to be taken into account when a partial realization is sought for an incompletely specified finite matrix sequence.

Partial Realizations in the Observer Design Context

The problem of realizing an incompletely specified finite matrix sequence with a system of the form (2.1) is discussed next, and then partial realization theory is applied to the specific situation of an observer design problem.

Consider an incompletely specified finite sequence $\{L_1, L_2, \ldots, L_{N_0}\}$ of $m \times r$ matrices together with its corresponding Hankel matrix $\mathcal{H}_{N_0 N_0}$. The dimension of the minimal realization of the sequence is given by the lowest rank that $\mathcal{H}_{N_0 N_0}$ can possibly attain after the unspecified elements are replaced by (appropriately chosen) numbers, and a minimal realization can be obtained by applying any one of several realization algorithms available in the literature to the full Hankel array $\mathcal{H}_{N_0 N_0}$. The results of Chapter 2 show that a more convenient and efficient way is to identify the recurrence relations (2.22) that hold for the columns of $\mathcal{H}_{N_0 N_0}$, as done in Algorithm (2.78).

For a completely specified finite matrix sequence, Step 1 of Algorithm (2.78) gives the dimension of the minimal realization and the two sets of invariants $\{\mu_{iM}\}$ and $\{\underline{\ell}_{jk}\}$. However, when the selection procedure of Step 1 of Algorithm (2.78) is applied to the columns of the Hankel array corresponding to an incompletely specified finite matrix sequence, the question of linear dependence of a column on the preceding columns depends on the specific values substituted for some of the unspecified elements. This implies that the set of minimal controllability indices of an incompletely specified finite matrix sequence may not be unique. On the other hand, assigning numerical values to the unspecified elements without a rule (such as the recurrence relations (2.22)) is not easy to do, may yield a non-minimal realization, and usually wastes degrees of freedom for the locations of the realization poles.

One approach to solve the problem is to write the recurrence relations (2.22) for every candidate set of minimal controllability indices and solve for the coefficients. This is not easily accomplished either because these equations may include products of the unspecified elements in the matrix sequence and the unknown coefficients $\{\gamma_{ijk}\}$, thus making the equations nonlinear. All of these formidable complications can be avoided completely if the realization problem is considered via the dual approach.

Observe that if $\underline{\ell}_{jk}$ is unspecified, then $\underline{\ell}_{jk+1},\ldots,\underline{\ell}_{jN_0}$ are also unspecified. Consequently, the unspecified elements in the columns of $\mathcal{H}_{N_0 N_0}$ occur always in the last positons. So, if the rows rather than the columns of $\mathcal{H}_{N_0 N_0}$ are examined for linear dependencies, then the unspecified elements are treated exactly as before. In conclusion, the

observability indices of the minimal realization of an incompletely specified finite matrix sequence are unique, while the minimal controllability indices are not, and a simple way to realize such a sequence is using the dual of the corresponding algorithm in Chapters 2 and 3. Notice that if rows instead of columns of the L_i matrices were unspecified, then the algorithms of the preceding chapters should be used directly (not in dual form).

It is difficult to carry out the preceding discussions in a more concrete form without referring to a specific example.

(4.41) Example. It is desired to obtain the minimal realization (in its most general form) of the following partial sequence

$$L_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad L_2 = \begin{bmatrix} \xi_{121} & 0 \\ \xi_{122} & 0 \\ \xi_{123} & 1 \end{bmatrix} \qquad L_3 = \begin{bmatrix} \xi_{131} & -1 \\ \xi_{132} & 1 \\ \xi_{133} & 2 \end{bmatrix}$$

The variables $\{\xi_{1jk}\}$ denote the elements of the unspecified columns $\underline{\ell}_{12} = \underline{\xi}_{12}$ and $\underline{\ell}_{13} = \underline{\xi}_{13}$. This sequence could have been obtained from the specifications of a minimal order observer design problem for a sixth-order plant with three inputs and two outputs, and observability indices $\nu_1 = 2$, $\nu_2 = 4$.

The Hankel array for the given sequence is

$$\mathcal{H}_{33} = \begin{bmatrix} 1 & 0 & \xi_{121} & 0 & \xi_{131} & -1 \\ 0 & 1 & \xi_{122} & 0 & \xi_{132} & 1 \\ 0 & 0 & \xi_{123} & 1 & \xi_{133} & 2 \\ \xi_{121} & 0 & \xi_{131} & -1 & * & * \\ \xi_{122} & 0 & \xi_{132} & 1 & * & * \\ \xi_{123} & 1 & \xi_{133} & 2 & * & * \\ \xi_{131} & -1 & * & * & * & * \\ \xi_{132} & 1 & * & * & * & * \\ \xi_{133} & 2 & * & * & * & * \end{bmatrix}$$

Examining the columns of $\mathcal{H}_{33}$ it is found that the minimal rank can be as low as three if $\underline{\xi}_{12}$ and $\underline{\xi}_{13}$ are chosen appropriately. But it is also found that such a choice is not unique; if $\xi_{123} = 0$, then $\mu_1 = 1$ and $\mu_2 = 2$, while if $\xi_{123} \neq 0$, then $\mu_1 = 2$ and $\mu_2 = 1$. Consider first the choice of structure $\mu_1 = 1$ and $\mu_2 = 2$. The equations that must be solved are given by

$$\begin{bmatrix} \xi_{121} \\ \xi_{122} \\ 0 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{111} \\ \gamma_{121} \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{211} \\ \gamma_{221} \\ \gamma_{222} \end{bmatrix}$$

The realization is immediately obtained as the following triple

$$A = \begin{bmatrix} -\gamma_{111} & 0 & -1 \\ -\gamma_{121} & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where γ_{111} and γ_{121} are arbitrary. This fixes $\underline{\xi}_{12}$ and $\underline{\xi}_{13}$ to be

$$\underline{\xi}_{12}^{T} = [-\gamma_{111} \quad -\gamma_{121} \quad 0]$$

$$\underline{\xi}_{13}^{T} = [\gamma_{111}^{2} \quad \gamma_{111}\gamma_{121} \quad -\gamma_{121}]$$

It can be easily shown that this realization has two arbitrary poles while the third pole is always unstable.

Consider now the choice of structure $\mu_1 = 2$ and $\mu_2 = 1$. The pertinent equations to be solved are

$$\begin{bmatrix} \xi_{131} \\ \xi_{132} \\ \xi_{133} \end{bmatrix} = - \begin{bmatrix} 1 & \xi_{121} & 0 \\ 0 & \xi_{122} & 1 \\ 0 & \xi_{123} & 0 \end{bmatrix} \begin{bmatrix} \gamma_{111} \\ \gamma_{112} \\ \gamma_{121} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 & \xi_{121} & 0 \\ 0 & \xi_{122} & 1 \\ 0 & \xi_{123} & 0 \end{bmatrix} \begin{bmatrix} \gamma_{211} \\ \gamma_{212} \\ \gamma_{221} \end{bmatrix}$$

The realization corresponding to this structure is given by the triple

$$A = \begin{bmatrix} 0 & -\gamma_{111} & -\gamma_{211} \\ 1 & -\gamma_{112} & -\gamma_{212} \\ 0 & -\gamma_{121} & -\gamma_{221} \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & \xi_{121} & 0 \\ 0 & \xi_{122} & 1 \\ 0 & \xi_{123} & 0 \end{bmatrix}$$

where

$$\gamma_{111} = \frac{\xi_{121}\xi_{133}}{\xi_{123}} - \xi_{131} \qquad \gamma_{211} = \frac{\xi_{121}}{\xi_{123}}$$

$$\gamma_{121} = \frac{\xi_{122}\xi_{133}}{\xi_{123}} - \xi_{132} \qquad \gamma_{212} = -\frac{1}{\xi_{123}}$$

$$\gamma_{112} = -\frac{\xi_{133}}{\xi_{123}} \qquad \gamma_{221} = \frac{\xi_{122}}{\xi_{123}}$$

with $\xi_{123} \neq 0$ and the remaining ξ_{1jk} elements arbitrary. It can also be shown that the available degrees of freedom are sufficient to guarantee arbitrary poles for this realization.

Consider now the dual of the realization procedure in Algorithm (2.78). Inspecting the rows of $\mathcal{H}_{33}$ it is noticed that the dimension of the minimal realization is equal to three and $\delta_{1M} = \delta_{2M} = \delta_{3M} = 1$. The pertinent equations to be solved in this case are

$$[0 \quad -1] = -[\theta_{111} \quad \theta_{121} \quad \theta_{131}] \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[0 \quad 1] = -[\theta_{211} \quad \theta_{221} \quad \theta_{231}] \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[1 \quad 2] = -[\theta_{311} \quad \theta_{321} \quad \theta_{331}] \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the realization is given in its most general form by the triple

$$A = \begin{bmatrix} -\theta_{111} & 0 & -1 \\ -\theta_{211} & 0 & 1 \\ -\theta_{311} & 1 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where θ_{111}, θ_{211}, and θ_{311} are arbitrary. This fixes $\underline{\xi}_{12}$ and $\underline{\xi}_{13}$ to be

$$\underline{\xi}_{12}^T = [-\theta_{111} \quad -\theta_{211} \quad -\theta_{311}]$$

$$\underline{\xi}_{13}^T = [\theta_{111}^2+\theta_{311} \quad \theta_{111}\theta_{211}+\theta_{311} \quad \theta_{111}\theta_{311}-\theta_{211}-2\theta_{311}]$$

It is easy to show that the realization poles are completely arbitrary.

The controllability matrix for this realization is

$$\hat{C} = [Q \quad AQ] = \begin{bmatrix} 1 & 0 & -\theta_{111} & 0 \\ 0 & 1 & -\theta_{211} & 0 \\ 0 & 0 & -\theta_{311} & 1 \end{bmatrix}$$

As before, the controllability indices are {2,1} if $\theta_{311} \neq 0$, or {1,2} if $\theta_{311} = 0$.

Notice that the dual procedure has given the same number of degrees of freedom for the positions of the realization poles and the same information about the sets of minimal controllability indices, while requiring a considerably smaller amount of effort and bookkeeping. ∇

Several of the partial realization theory results of Chapter 3 are examined next in the observer context. The following remark opens the discussion.

(4.42) Remark. Consider a triple $(A,Q,D)_p$ which represents a partial realization of a finite matrix sequence that was defined as a result of an observer design problem. Then the observability indices of the pair $(A,D)_p$ are also the observability indices of the observer system, but the controllability indices of the pair $(A,Q)_p$ are not the controllability indices of the observer system. The controllability indices of the observer are those of the pair $(A,[B \quad C])_p$. There is no apparent reason to believe that the controllability indices of $(A,Q)_p$ and $(A,[B \quad C])_p$ are related in any meaningful way. ∇

To design a minimal order observer for a given feedback control law and plant using partial realization theory, the appropriate algorithm to use is the dual of Algorithm (3.42) (the minimal partial stable

realization algorithm). Of all the results given in Chapter 3, those related to the minimal partial stable realization problem are the weakest. Correspondingly, the minimal order observer design problem has the weakest results. For example, it seems to be practically impossible to know the dimension of the minimal order observer without obtaining the triple $(A,Q,D)_p$ first and calculating the eigenvalues of A. Only some bounds for the dimension of the minimal order observer can be stated. This is discussed in the following remark.

(4.43) Remark. A bound for the dimension of the minimal order observer has been given in Lemma (4.18). Viewed from the partial realization formulation of an observer design problem, it is apparent that bound (4.19) is not too strong in general. Another bound can be derived from Theorem (3.31) and its vector sequence counterparts, Theorems (3.27) and (3.28). It consists of obtaining the smallest value of p for which conditions (3.31a) and (3.31b) are satisfied for the partial sequence formed from the specifications of a minimal order observer design problem. This bound is also considerably weak because these two conditions are necessary but not sufficient for a partial realization to be stable. The strongest bound readily available is given by the minimal rank of $\mathcal{H}_{N_0 N_0}$. That is, the dimension p of a minimal order observer satisfies the following inequality

$$p \geq \rho(\mathcal{H}_{N_0 N_0}) \tag{4.44}$$

where $\rho(\mathcal{H}_{N_0 N_0})$ is calculated using the dual form of Step 1 of Algorithm (2.78). This bound is actually satisfied in a large number of cases. A good rule to follow is to compute the bound (4.44) and verify whether (3.31a) and (3.31b) are satisfied or not for this value of p.

Rothschild and Grammaticos (1971) have given (without proof) the following bound for the dimension of a minimal order observer for a given feedback control law and plant

$$p \geq \frac{m(n - r)}{m + r} \tag{4.45}$$

The derivation of this bound is supposed to have been made under the assumptions that matrix A is specified and that it has no eigenvalues in common with matrix F. It is intuitively clear that some degrees of freedom are lost by introducing these two assumptions and, consequently, (4.44) should be a stronger bound in many cases.

Based on the discussions in Chapter 3 with respect to the value of p_S, it seems to be quite difficult to obtain a stronger bound than (4.44). On the other hand, the bound (4.44) requires considerably more computations than the other three bounds mentioned here. ∇

The appropriate algorithm to use in designing an intermediate order observer using partial realization theory is the dual of Algorithm (3.36). As in the case of minimal order observers, not much can be specifically said about intermediate order observers. But there are two significant points that should be discussed.

(4.46) Remark. It is possible that intermediate order observers of a specific dimension do not exist at all for a given feedback control law and plant. This is a consequence of the fact that a partial realization of a specific dimension may not exist for a given finite matrix sequence (see Remark (3.35)). A further complication is that there is no way of knowing beforehand whether such a condition arises or not in a given problem. ∇

(4.47) Remark. Given that an intermediate order observer of a specific dimension exists, the number of arbitrary poles it has is not easily

determined in general. A good estimate consists of the number of arbitrary $\{\theta_{ijk}\}$ (or $\{\gamma_{ijk}\}$) parameters in A. ∇

The stronger results available for the minimal partial arbitrary realization problem correspondingly yield strong results (in the form of necessary and/or sufficient conditions) for the problem of designing minimal order observers with arbitrary poles. The most significant results that can be adopted from Chapter 3 through Theorem (4.34) are stated in the following theorems.

(4.48) Theorem. A p-dimensional observer which estimates the feedback control law $K\underline{x}$ for a single output system (4.1) has arbitrary poles if and only if

(4.49) $$p \geqq \nu - 1 = n - 1$$

Proof. Follows readily from Theorems (4.34) and (3.10). ∇

(4.50) Theorem. A p-dimensional observer which estimates the feedback control law $\underline{k}^T\underline{x}$ for a multiple output system (4.1) has arbitrary poles if and only if

(4.51) $$p \geqq \nu_\alpha - 1$$

where $\nu_\alpha - 1 \leqq \nu - 1$ is the length of the longest non-zero chain of $\underline{k}_1^T$.

Proof. In view of Theorem (4.34), it suffices to show that a triple $(A,Q,\underline{d}^T)_p$ which realizes a finite sequence of r-dimensional row vectors formed from a given (n-r)-dimensional row vector $\underline{k}_1^T$ has aribtrary poles if and only if (4.51) is satisfied. In this proof it is assumed that the observability indices of $(F,H)_n$ are arranged in non-increasing fashion, and the last $r - \alpha + 1$ indices are considered to be equal to each other; that is, $\nu = \nu_1 \geqq \nu_2 \geqq \cdots \geqq \nu_{\alpha-1} \geqq \nu_\alpha = \nu_{\alpha+1} = \cdots = \nu_r$.

It is well known (Kalman, 1971a) that no loss of generality is involved in assuming the observability indices of the plant to be arranged in non-increasing fashion. The assumption concerning the equality of the last $r - \alpha + 1$ observability indices is not necessary for the proof and can be easily removed. Both of these assumptions are introduced exclusively for notational simplicity.

Sufficiency. Suppose that the first $\alpha - 1$ (for some integer $1 \leqq \alpha \leqq r$) chains of $\underline{k}_1^T$ are zero; that is,

$$k_{11} = k_{12} = \ldots = k_{1\upsilon_1} = k_{21} = \ldots = k_{\alpha-1\upsilon_{\alpha-1}} = 0$$

This makes $\upsilon_\alpha = \nu_\alpha - 1$ be the length of the longest non-zero chain of $\underline{k}_1^T$. The corresponding row vector sequence formed according to Theorem (4.34) is given by

$$\begin{aligned}
\underline{\zeta}_1^T &= [0 \ . \ . \ . \ 0 \quad k_{\alpha 1} \ . \ . \ . \ k_{r1}] \\
\underline{\zeta}_2^T &= [0 \ . \ . \ . \ 0 \quad k_{\alpha 2} \ . \ . \ . \ k_{r2}] \\
&\ \vdots \\
\underline{\zeta}_{\upsilon_\alpha}^T &= [0 \ . \ . \ . \ 0 \quad k_{\alpha\upsilon_\alpha} \ . \ . \ . \ k_{r\upsilon_\alpha}] \\
\underline{\zeta}_i^T &= [0 \ . \ . \ . \ 0 \quad * \ . \ . \ . \ *] \qquad i = \upsilon_\alpha+1,\ldots,N_0
\end{aligned} \tag{4.52}$$

The extension sequence is unspecified, so the first $\alpha - 1$ elements of every vector in the extension sequence can be set equal to zero. That is, the extension sequence is chosen in the form

$$\underline{\zeta}_i^T = [0 \ . \ . \ . \ 0 \quad * \ . \ . \ . \ *] \qquad i = N_0+1,\ldots \tag{4.53}$$

Consider now the dual of equation (3.11) with the vectors in (4.52) and (4.53). This is the following equation

$$\underline{\zeta}_{p+1+\tau}^T = -\sum_{k=1}^{p} \beta_k \underline{\zeta}_{k+\tau}^T \qquad \tau = 0,1,2,\ldots \tag{4.54}$$

Observe that for the first $\alpha - 1$ elements of the vectors $\{\underline{\zeta}_1^T, \underline{\zeta}_2^T, \ldots\}$ equation (4.54) reads $0 = 0$ with any $p > 0$ and any set of p coefficients $\{\beta_k\}$. Observe also that for $p \geq \upsilon_\alpha = \nu_\alpha - 1$ the last $r - \alpha + 1$ elements of the vectors in the right-hand-side of (4.54) are unspecified. Thus, if (4.51) is satisfied, the unspecified (non-zero) elements of the extension sequence can be chosen to satisfy (4.54) for any given set of p coefficients $\{\beta_k\}$. Finally, the realization is constructed in the dual of form (3.12)-(3.14).

Necessity. This part of the proof is omitted because it is the exact dual of the necessity part of the proof of Theorem (3.10). ∇

These two theorems sharpen the available results in the problem of designing minimal order observers with arbitrary poles for single output plants and for multiple output plants where $\underline{k}^T\underline{x}$ is to be estimated. The sufficiency part of Theorem (4.48) is well-known (Luenberger, 1964, 1966, 1971), but the necessity part has not been proven elsewhere. Williamson (1970) recently demonstrated the sufficiency part of Theorem (4.50), but not the necessity part. Theorem (4.50) is an excellent example of the fact that the occurrence of a null chain in K_1 can lead to a reduction in the order of the required observer.

The corresponding results for multiple input/output plants are given in the following two dual theorems.

(4.55) Theorem. If a quintuple $(A,B,C,D,E)_p$ defines an observer with arbitrary poles for the feedback control law $K\underline{x}$ of a multiple input/output plant $(F,G,H)_n$, then its dimension and the controllability indices $\{\mu_i\}$ of the associated matrix pair $(A,Q)_p$ satisfy the following constraints:

a) $\mu_{iM} \leqq \mu_i \leqq \nu_i - 1$ $\qquad i = 1,2,\ldots,r$

b) $p < \sigma - \sum_{i=1}^{r} \rho(\hat{M}_i)$

c) $p \leqq \min(n-r, m(\nu-1))$

where $\{\mu_{iM}\}$ is the set of minimal controllability indices of the finite matrix sequence formed from K, $\{\nu_i\}$ is the set of observability indices of $(F,H)_n$, σ is defined by the set $\{\mu_i\}$ through (2.23) and (2.51), and $\hat{M}_i$ is defined in Step 2 of Algorithm (2.78).

Proof. The lower bound in condition (4.55a), condition (4.55b), and the bound $p \leqq m(\nu - 1) = mN_0$ follow from Theorems (4.34) and Corollary (3.23). The upper bound in condition (4.55a) and the bound $p \leqq n - r$ are obtained from well-known sufficiency results in the desing of observers with arbitrary poles (see, for example, Luenberger, 1964, 1966, 1971, or Gopinath, 1971). ∇

In view of Remark (4.42), the dual of Theorem (4.55) is different enough to deserve specific mention.

(4.56) Theorem. If a quintuple $(A,B,C,D,E)_p$ defines an observer with arbitrary poles for the feedback control law $K\underline{x}$ of a multiple input/output plant $(F,G,H)_n$, then its dimension and its observability indices $\{\delta_i\}$ satisfy the following constraints:

a) $\delta_{iM} \leqq \delta_i \leqq \nu - 1$ $\qquad i = 1,2,\ldots,m$

b) $p \leqq \kappa - \sum_{i=1}^{m} \rho(\hat{R}_i)$

c) $p \leqq \min(n-r, m(\nu-1))$

where $\{\delta_{iM}\}$ is the set of minimal observability indices of the finite matrix sequence formed from K, ν is the observability index of $(F,H)_n$, and κ and the matrices $\hat{R}_i$ are the dual of σ and the matrices $\hat{M}_i$, respectively.

Proof. Follows by duality. ∇

These two theorems provide a set of necessary conditions and sufficient conditions for a minimal order arbitrary observer to estimate a given feedback control law of a specified plant. Conditions (4.55b), (4.56b), and the lower bounds on (4.55a) and (4.56a) are necessary conditions, while (4.55c), (4.56c), and the upper bound on (4.55a) and (4.56a) are sufficient conditions. The sufficient conditions are well known (Luenberger, 1964, 1966, 1971; Jameson and Rothschild, 1971; Wolovich, 1973a), but the necessary conditions stated in the theorems are not available elsewhere. This makes the contribution of the theorems be significant because sufficient conditions are generally more conservative than necessary conditions in the following sense.

Consider the problem solved in Example (3.43). Suppose that the partial sequence was obtained as a formulation of a minimal partial realization problem from a minimal order observer design problem for a two-input, two-output, tenth-order system with observability indices $\nu_1 = \nu_2 = 5$. The sufficient conditions of Theorems (4.55) and (4.56) give $p = 8$, $\mu_1 = \mu_2 = 4$, and $\delta_1 = \delta_2 = 4$ for the dimension and structure, respectively, of an arbitrary realization. However, it can be shown that all partial realizations of dimension equal to 6 have arbitrary poles and observability indices given either by $\{4,2\}$ or $\{3,3\}$, depending on the specific values chosen for the arbitrary $\{\gamma_{ijk}\}$ parameters.

It is evident that Theorems (4.55) and (4.56) are not as strong as Theorems (4.48) and (4.50). However, there are reasons to believe that this is not necessarily a deficiency. It is strongly suspected that the conditions listed in these theorems are the best one can do without considering a detailed analysis of the problem for all possible feedback control laws for a given system. The example given at the end of this chapter lends support to this claim (see also Remark (4.82)).

Keeping in mind the results discussed in this chapter, the algorithms outlined in Chapter 3 can conceivably be used to obtain all intermediate order observers of a specified dimension, all minimal order observers, and all minimal order arbitrary observers for a given feedback control law and plant. Another problem that can be studied is that of obtaining the class of feedback control laws that can be observed by a given observer for a given plant, or by an observer of specified dimension and output structure but unspecified dynamics for a given plant. Other similar (and perhaps less ambitious) problems could also be easily proposed. An example given in the sequel considers some of these problems for a specific case.

The partial realization approach to observer design also serves to provide a unified framework for studying and evaluating other observer design procedures. This is considered next.

Partial Realization Equivalent of Other Observer Designs

The partial realization algorithms corresponding to some of the currently available (full-state and intermediate order) observer design methods are studied in this section. Specifically, the full-state observer designs of Luenberger (1966), Gopinath (1971), and Munro (1973,

1974) are considered, along with the intermediate order observer designs (for a given feedback control law $K\underline{x}$) of Fortmann and Williamson (1972) and Murdoch (1974).

There are several other (both full-state and intermediate order) observer design procedures that may correspond with a partial realization procedure. Notably among these other procedures are the full-state observer design of Bongiorno and Youla (1968), the full-state observer design (via the transfer function matrix approach) of Retallack (1970), the intermediate order (of dimension equal to $m(\nu - 1)$) observer design of Jameson and Rothschild (1971), the intermediate order (of dimension equal to $m(\nu - 1)$) observer design (via the matrix-fraction description approach) of Wolovich (1973a), and the minimal order (not always stable, though) observer design (via the matrix-fraction description approach) of Wang and Davison (1973). The correspondence (if any at all) of these observer design techniques with a partial realization formulation of the problem is not obvious and remains to be investigated.

(4.57) Full-state observer design of Luenberger (1966) and Munro (1973, 1974). This observer design relates to a simple (but non-minimal) partial realization algorithm. The main idea involved in the observer design is the following. Consider (without loss of generality) the plant (4.1) to be given with the pair $(F,H)_n$ in the form (4.26)-(4.27), with

(4.58) $$\underline{f}_i^T = [\underline{f}_{i1}^T \quad \underline{f}_{i2}^T \; . \; . \; . \; \underline{f}_{ir}^T] \qquad i = 1,2,\ldots,r$$

where $\underline{f}_{ij}$ is a ν_j-dimensional vector. In this form, the plant may be considered as a parallel combination of single output subsystems coupled in both directions by their respective outputs. Since all the inputs to every subsystem are available, then subobservers can be designed to estimate the state of every one of the single output subsystems, and the

overall observer is the (uncoupled) parallel combination of all the sub-observers. In equation form, the observer parameters are given by

(4.59a) $$T = \mathrm{diag}(T_i)$$

(4.59b) $$T_i = [\; I_{\nu_i - 1} \quad -\underline{\gamma}_{ii}] \qquad i = 1,2,\ldots,r$$

(4.59c) $$\underline{\gamma}_{ii}^T = [\gamma_{ii1} \quad \gamma_{ii2} \;\cdot\;\cdot\;\cdot\; \gamma_{ii\nu_i - 1}] \qquad i = 1,2,\ldots,r$$

(4.60a) $$A = \mathrm{diag}(A_{ii})$$

(4.60b) $$A_{ii} = \left[\begin{array}{c|c} \underline{0}_{\nu_i-2}^T & \\ \hline I_{\nu_i-2} & \end{array} \; -\underline{\gamma}_{ii}\right] \qquad i = 1,2,\ldots,r$$

(4.61) $$B = TG$$

(4.62a) $$C = \begin{bmatrix} \underline{c}_{11} & \underline{c}_{12} & \cdots & \underline{c}_{1r} \\ \underline{c}_{21} & \underline{c}_{22} & \cdots & \underline{c}_{2r} \\ \vdots & \vdots & & \vdots \\ \underline{c}_{r1} & \underline{c}_{r2} & \cdots & \underline{c}_{rr} \end{bmatrix}$$

(4.62b) $$\underline{c}_{ii} = -T_i\underline{f}_{ii} - A_{ii}\underline{\gamma}_{ii}$$

(4.62c) $$\underline{c}_{ij} = -\; T_i\underline{f}_{ji}$$

(4.63a) $$D = [D_1 \quad D_2 \;\cdot\;\cdot\;\cdot\; D_r]$$

(4.63b) $$D_i = [\underline{i}_{\hat{\nu}_{i-1}+1} \quad \underline{i}_{\hat{\nu}_{i-1}+2} \;\cdot\;\cdot\;\cdot\; \underline{i}_{\hat{\nu}_i - 1}]$$

(4.64a) $$E = [\underline{e}_1 \quad \underline{e}_2 \;\cdot\;\cdot\;\cdot\; \underline{e}_r]$$

(4.64b) $$\underline{e}_i = -\; D_i\underline{\gamma}_{ii} + \underline{i}_{\hat{\nu}_i}$$

where $\hat{\nu}_i = \nu_{i-1} + \nu_i$ for $1 \leqq i \leqq r$ with $\hat{\nu}_0 = 0$, and the $(\nu_i - 1)$-dimensional vectors $\{\underline{\gamma}_{ii}\}$ completely arbitrary. Since the $\{\underline{\gamma}_{ii}\}$ vectors are completely arbitrary then the n - r observer poles are completely arbitrary, with

the (possible) exception that some poles may assume only real values (for example, if $\nu_1 - 1 = \nu_2 - 1 = 3$, then it is not possible for the six observer poles to be three complex conjugate pairs; two poles must be real). Luenberger (1966) and Munro (1973, 1974) further impose the condition that there be no poles in common between the plant and observer in order for matrix T to be unique (see also Luenberger, 1964). Wonham (1970) was the first to show that these two restrictions on the values of the observer poles are unnecessary, and that the non-uniqueness of T can be handled favorably.

The partial realization algorithm corresponding to this observer design is the procedure outlined in the proof of Theorem (3.22), modified to have the controllability indices of the pair $(A,Q)_{n-r}$ be the set $\{\mu_i : \mu_i = \nu_i - 1\}$ instead of the set $\{\mu_i : \mu_i = N_0 = \nu - 1\}$. This obvious modification is necessary because every one of the observability indices of $(F,H)_n$ is not generally equal to ν.

From the point of view of partial realization theory, the restrictions on the values of the observer poles are easily removed. This is done by choosing matrix A in the form (2.55) rather than (4.60). Only one of the equations (4.59), (4.61)-(4.64) needs to be correspondingly modified: (4.62c) is replaced by

(4.62c') $$\underline{c}_{ij} = - T_i \underline{f}_{ji} - A_{ij} \underline{\gamma}_{jj}$$

The matrix Q for the realization corresponding to this observer design is

(4.65) $$Q = [\underline{i}_1 \quad \underline{i}_{\mu_1+1} \quad \underline{i}_{\mu_1+\mu_2+1} \quad \cdots \quad \underline{i}_{n-r-\mu_r}]$$

In an actual design, the gain in degrees of freedom for the values of the observer poles has to be properly weighed against the increase in complexity of the observer.

∇

(4.66) Full-state observer design of Gopinath (1971). The basic ideas involved in this observer design are the following. Consider the plant (4.1) with the defined partitions, and let $K = I_n$. Rewrite (4.8c) as

(4.67) $$I_n = [D \quad E] \begin{bmatrix} T_1 & T_2 \\ 0_{r,n-r} & I_r \end{bmatrix}$$

T_1 is dimensioned $(n-r)\times(n-r)$, so it is obvious that it must be nonsingular. The simplest choice is $T_1 = I_{n-r}$. Then (4.8a) and (4.14)-(4.17) become

(4.68) $$A = F_{11} + T_2F_{21}$$

(4.69) $$B = G_1 + T_2G_2$$

(4.70) $$C = F_{12} + T_2F_{22} - F_{11}T_2 - T_2F_{21}T_2$$

(4.71) $$D^T = [I_{n-r} \quad 0_{n-r,r}]$$

(4.72) $$E^T = [-T_2^T \quad I_r]$$

Observe that T_2 is the only unknown in the right-hand side of these equations. The criterion to select T_2 is discussed next.

Recall (Lemma (4.22)) that $(F_{11},F_{21})_{n-r}$ is a completely observable pair; it follows by duality that $(F_{11}^T,F_{21}^T)_{n-r}$ is a completely controllable pair. Then, matrix T_2^T can be chosen so that the eigenvalues of

(4.73) $$A^T = F_{11}^T + F_{21}^TT_2^T$$

are given by any specified set of n - r complex eigenvalues (with complex numbers occurring in conjugate pairs). This result is due to Wonham (1967). Kalman (1971a) has further shown that the choice of T_2^T to obtain a specified set of eigenvalues for A^T is not unique, except when F_{21}^T is a column vector. The eigenvalues of A are equal to the eigenvalues of A^T, so the design allows completely arbitrary observer poles. The partial realization algorithm corresponding to this observer design is discussed next.

Label the columns of $T_1 = I_{n-r}$ as in (4.30). It follows from (4.33) that the matrix Q which corresponds to this observer design is given by (4.65) also. The form of matrix Λ depends on the particular pole placement procedure used to compute T_2. This means that the pole placement procedure applied to (4.73) determines the partial realization algorithm applied to the corresponding realization theory formulation of the observer problem. The following two correspondences can be made immediately. The pole placement algorithm of Kalman (1971a) corresponds to Algorithm (2.78), and the pole placement algorithm of Wonham (1967) corresponds to the partial realization algorithm of Ackermann (1972) (which is outlined, in dual context, in Remark (2.83)). ∇

(4.74) Intermediate order observer design of Fortmann and Williamson (1972). This is a low order observer design procedure (for vector linear functions of the state) based on the same concept as (4.57). Consider the plant (4.1) with the pair $(F,H)_n$ given in the form (4.26)-(4.27). Partition the state $\underline{x}$ and matrix K corresponding to the subsystems in (4.1). That is,

(4.75) $$\underline{x}^T = [\underline{x}_1^T \quad \underline{x}_2^T \quad . \; . \; . \quad \underline{x}_r^T]$$

(4.76) $$K = [\tilde{K}_1 \quad \tilde{K}_2 \; . \; . \; . \; \tilde{K}_r]$$

where $\underline{x}_i$ is $\nu_i \times 1$ and $\tilde{K}_i$ is $m \times \nu_i$. It follows that

(4.77) $$K\underline{x} = \tilde{K}_1\underline{x}_1 + \tilde{K}_2\underline{x}_2 + . \; . \; . + \tilde{K}_r\underline{x}_r$$

Fortmann and Williamson (1972) estimate each $\tilde{K}_i\underline{x}_i$ with a subobserver, and add the outputs of the r subobservers to obtain an asymptotic estimate of $K\underline{x}$. The subobservers they design for the feedback control law $K_i\underline{x}_i$ of each multiple input, single output subsystem are minimal order subobservers. But it does not follow (except in unusual circumstances) that the overall observer formed by the uncoupled parallel combination

of the subobservers is a minimal order observer for the total feedback control law $K\underline{x}$ of the multiple input/output plant (4.1).

The partial realization algorithm corresponding to this particular observer design is the stable (rather than arbitrary) version of the algorithm outlined in the proof of Theorem (3.22). ∇

(4.78) Intermediate order observer design of Murdoch (1974). An outline of this observer design procedure is as follows. Consider the plant (4.1) and the feedback control law (4.2). Define K in terms of its rows as

$$K = \begin{bmatrix} \underline{k}_1^T \\ \underline{k}_2^T \\ \cdot \\ \cdot \\ \cdot \\ \underline{k}_m^T \end{bmatrix} \tag{4.79}$$

Design a subobserver of dimension $\nu - 1$ and arbitrary poles to observe the first linear function, $\underline{k}_1^T\underline{x}$, using any one of several techniques (Luenberger, 1966; Williamson, 1970; Wonham and Morse, 1972; Wolovich, 1973a; Murdoch, 1973; Wang and Davison, 1973; Roman <u>et al.</u>, 1973). Let $\underline{z}_1$ and w_1 be the state and output, respectively, of this subobserver, and let $\tilde{T}_1$ be the linear transformation of $\underline{x}$ to which $\underline{z}_1$ is asymptotic; that is, $\underline{z}_1 \to \tilde{T}_1\underline{x}$ and $w_1 \to \underline{k}_1^T\underline{x}$ as $t \to \infty$.

In the steady-state, $\underline{z}_1 = \tilde{T}_1\underline{x}$. This can be used advantageously to design subobservers for the remaining scalar linear functions as follows. Define $\tilde{H}_2$ to be the following $(r+\nu-1)\times n$ matrix

$$\tilde{H}_2 = \begin{bmatrix} H \\ \tilde{T}_1 \end{bmatrix}$$

Both $\underline{z}_1$ and $\underline{y}$ provide information about $\underline{x}$. Further, assuming the largest chain in $\underline{k}_1^T$ is non-zero, the information provided by $\underline{z}_1$ is linearly independent of the information contained in $\underline{y}$ (see Theorem (4.50)). This implies $\tilde{H}_2$ has full rank. (If the largest chain in $\underline{k}_1^T$ is a null chain, then $r < \rho(\tilde{H}_2) \leqq r + \nu - 1$ because some rows of $\tilde{T}_1$ may be a linear combination of the other rows in $\tilde{H}_2$.) Then the information available in $\underline{z}_1$ and in $\underline{y}$ is used to observe $\underline{k}_2^T\underline{x}$ with a subobserver of dimension $\tilde{\nu}_2 - 1$ and arbitrary poles, where $\tilde{\nu}_2$ is the observability index of the pair $(F,\tilde{H}_2)_n$.

The remaining scalar linear functions are observed using the same principle. Let $\tilde{H}_i$ be the following $(r+\nu+\tilde{\nu}_2+\ldots+\tilde{\nu}_{i-1}+i-1)\times n$ matrix

$$\tilde{H}_i = \begin{bmatrix} H \\ \tilde{T}_1 \\ \tilde{T}_2 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{T}_{i-1} \end{bmatrix} \tag{4.80}$$

Generally, $\rho(\tilde{H}_i) > \rho(\tilde{H}_{i-1})$. It follows that $\tilde{\nu}_i \leqq \tilde{\nu}_{i-1}$, where $\tilde{\nu}_j$ is the observability index of the pair $(F,\tilde{H}_j)_n$. Then the information available in $\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_{i-1}$, and $\underline{y}$ is used to design the ith subobserver (which estimates $\underline{k}_i^T\underline{x}$) of dimension $\tilde{\nu}_i - 1$ and arbitrary poles.

The overall observer has dimension $p = \nu + \tilde{\nu}_2 + \ldots + \tilde{\nu}_m = m \leqq m(\nu - 1)$, consists of a cascade of m subobservers coupled in the forward direction only, and has arbitrary poles (except that some may be constrained to be real). The overall observer system matrix, A, has (block) lower-triangular form and (practically) arbitrary eigenvalues. The coordinates in which the overall observer is represented depend on the procedures used to design the subobservers.

Suppose the subobservers are designed to be minimal order subobservers rather than minimal order arbitrary observers; then, it is evident that an observer designed as in (4.74) will generally have dimension larger than an observer designed following the above procedure. But whichever the case in question (whether minimal order or minimal order arbitrary observers), the design procedure of Murdoch (1974) is often non-minimal and less general than the observer design procedure proposed in this chapter.

Designing an intermediate order observer according to Murdoch (1974) is equivalent to using the algorithm of Ackermann (1972) to solve the partial realization problem formulated from the observer design problem. It has been shown (see Remark (2.83) and Example (2.85)) that a partial realization obtained using Ackermann's (1972) algorithm is often non-minimal and less general than a partial realization obtained using Algorithm (2.78). ∇

As previously mentioned, there are several other full-state and intermediate order observer design procedures that remain to be investigated in the partial realization theory context. Notably among these is the design algorithm proposed by Wang and Davison (1973). Their results are discussed briefly next (see also Chapter 1).

Wang and Davison (1973) consider the problem of observing vector linear functions of the state with a minimal order observer, and approach the problem from the matrix-fraction rather than the state-space point of view. Their procedure gives a minimal order system which satisfies all the requirements of an observer, except possibly stability. This agrees with the results given here because the minimal partial realization of the finite matrix sequence formed from the given observer design

problem may not be stable. Wang and Davison (1973) do not give a procedure to obtain a higher order but stable system in the cases where the minimal order system is unstable.

The relationships which exist between state-space and matrix-fraction descriptions (see Theorems (2.54) and (2.60)) suggest that there must be matrix-fraction description counterparts for the state-space description results of this dissertation. The work of Dickinson et al. (1974a, b) gives more support to this conjecture. Further research in this problem is needed.

The example given below illustrates the power and generality of the realization theory approach to solve an observer design problem. The dimension and number of arbitrary poles of a minimal order observer which estimates a class of feedback control laws for plants of dimension equal to five and a specified output structure are obtained in the example.

(4.81) Example. Consider any fifth-order, two-input, two-output plant which has observability indices $\nu_1 = 3$ and $\nu_2 = 2$, and suppose the plant is given in the coordinates (4.26)-(4.27). (A plant with observability indices $\nu_1 = 2$ and $\nu_2 = 3$ is converted to one with observability indices $\nu_1 = 3$ and $\nu_2 = 2$ by reordering the two outputs.) Consider also all full-rank feedback matrices K given in the same coordinates as the plant. It is desired to study the class of feedback control laws that can be estimated with a minimal order observer. Only full-rank feedback matrices are considered because otherwise the problem reduces to observing a scalar linear function of the state.

In view of Theorem (4.34) and Corollary (4.38), it is necessary to study only the possible partial realizations of a matrix sequence formed from the columns of K_1 (the first three columns of K). Following (4.35), label the elements of K_1 as follows

$$K_1 = [\underline{k}_{11} \quad \underline{k}_{12} \quad \underline{k}_{21}] = \begin{bmatrix} k_1 & k_5 & k_3 \\ k_2 & k_6 & k_4 \end{bmatrix}$$

The incompletely specified matrix sequence formed according to (4.36) is given by

$$L_1 = \begin{bmatrix} k_1 & k_3 \\ k_2 & k_4 \end{bmatrix} \qquad L_2 = \begin{bmatrix} k_5 & \xi_7 \\ k_6 & \xi_8 \end{bmatrix}$$

where ξ_7 and ξ_8 are unspecified elements; the corresponding Hankel array is

$$\mathcal{H}_{22} = \begin{bmatrix} k_1 & k_3 & k_5 & \xi_7 \\ k_2 & k_4 & k_6 & \xi_8 \\ k_5 & \xi_7 & * & * \\ k_6 & \xi_8 & * & * \end{bmatrix}$$

The dimension of the minimal order observer depends on the rank of $\mathcal{H}_{22}$.

To investigate the minimal rank of $\mathcal{H}_{22}$, the unspecified elements (ξ_7, ξ_8, and the asterisks)are neglected. The following observations can be made by inspection of the rows of $\mathcal{H}_{22}$. Since K_1 has full rank ($\rho(K_1) = 2$), the minimal rank of $\mathcal{H}_{22}$ cannot be less than two and cannot be more than three. Also, it will be equal to three if and only if $\underline{k}_{11} = \underline{0}$. The following conclusion can be reached immediately: the minimal order observer has dimension equal to three and arbitrary poles if $\underline{k}_{11} = \underline{0}$.

To proceed with the discussion, let $\underline{k}_{11} \neq \underline{0}$. Then the minimal partial realization of the sequence has dimension equal to two and (by inspection) it is given in its most general form by the triple

$$A = \begin{bmatrix} -\gamma_{111} & -\gamma_{211} \\ -\gamma_{121} & -\gamma_{221} \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} k_1 & k_3 \\ k_2 & k_4 \end{bmatrix}$$

where γ_{211} and γ_{221} are completely arbitrary, and

$$\begin{bmatrix} k_5 \\ k_6 \end{bmatrix} = - \begin{bmatrix} k_1 & k_3 \\ k_2 & k_4 \end{bmatrix} \begin{bmatrix} \gamma_{111} \\ \gamma_{121} \end{bmatrix}$$

The characteristic polynomial of A is given by

$$\beta(s) = s^2 + \beta_2 s + \beta_1$$

$$\beta(s) = s^2 + (\gamma_{111} + \gamma_{221})s + \gamma_{111}\gamma_{221} - \gamma_{121}\gamma_{211}$$

Notice that any two given coefficients β_1 and β_2 can be matched with an appropriate choice of γ_{211} and γ_{221} if and only if $\gamma_{121} \neq 0$. It follows immediately that the minimal order observer has dimension equal to two and arbitrary poles if and only if $\underline{k}_{11} \neq \underline{0}$ and $\underline{k}_{12}$ is linearly independent of $\underline{k}_{11}$.

It remains to consider the consequences of $\underline{k}_{12}$ being linearly dependent on $\underline{k}_{11}$. Let $\gamma_{121} = 0$ and $\gamma_{111} \neq 0$ (this insures $\underline{k}_{11} \neq \underline{0}$). Then

$$\beta_1 = \gamma_{111}\gamma_{221}$$
$$\beta_2 = \gamma_{111} + \gamma_{221}$$

For stability, it is necessary and sufficient that $\beta_1 > 0$ and $\beta_2 > 0$; this, in turn, implies $\gamma_{111} > 0$ and $\gamma_{221} > 0$. But γ_{221} is arbitrary, so stability is guaranteed if $\gamma_{111} > 0$.

This result is combined with the previous ones into the following statements.

a) The minimal order observer has dimension equal to three and arbitrary poles if and only if

i) $\underline{k}_{11} = \underline{0}$

or ii) $\underline{k}_{12} = -\gamma_{111}\underline{k}_{11}$ with $\gamma_{111} \leqq 0$.

b) The minimal order observer has dimension equal to two and arbitrary poles if and only if $\underline{k}_{11} \neq \underline{0}$ and $\underline{k}_{12}$ is linearly independent of $\underline{k}_{11}$.

c) The minimal order observer has dimension equal to two, one pole arbitrary (but real), and the other pole at $-\gamma_{111}$ if and only if $\underline{k}_{11} \neq \underline{0}$ and $\underline{k}_{12} = -\gamma_{111}\underline{k}_{11}$ with $\gamma_{111} > 0$.

It is emphasized that these statements are true given the triple $(F,G,H)_n$ and matrix K in the coordinates of Lemma (4.25).

The above statements characterize the classes of full-rank feedback matrices for the control law (4.2) that can be estimated with a minimal order observer for all fifth-order, two-input, two-output plants which have observability indices given by the set {3,2}. Almost quite as remarkable as this result itself is the ease with which it was obtained.∇

Example (4.81) illustrates some of the types of results that can be obtained by studying observer-plant relationships via partial realization theory. Of course, as the dimension of the plant increases and its output structure becomes more complex, the amount of work involved increases at a large rate and it may not be possible to be as conclusive as in Example (4.81), but it may still be possible to obtain results in terms of necessary conditions and sufficient conditions.

(4.82) Remark. A significant point is brought to light in Example (4.81). Consider the necessary conditions and the sufficient conditions of Theorem (4.55). For this example, these conditions imply

a) $1 \leq \mu_1 \leq 2$

b) $\mu_2 = 1$

c) $2 \leq p \leq 3$

But is does not seem to be possible to make more definite conclusions (regarding the dimension and structure of the minimal order observer) unless K_1 is specified. This suggests that (for multiple input/output plants) there may not exist a stronger set of conditions than those listed in Theorem (4.55). It is interesting to notice that the bounds for the dimension of the minimal order observer listed in Remark (4.43) give $p \geq 2$. ∇

The results presented in this chapter extend the available theory considerably, but there still remain several details to be investigated further and various possible extensions to be considered. This is discussed more amply in the following chapter.

CHAPTER 5
CONCLUDING REMARKS

A summary of the contributions of this dissertation and suggestions for future research are given below.

Summary

This dissertation contains contributions to both realization theory and observer design theory. The major contributions are outlined next by chapters.

The problem of obtaining a complete system of invariants for the equivalence classes defined by similarity transformations on the state-space of linear, constant systems is discussed in Chapter 2. A corresponding set of canonical forms is defined. This leads naturally to the formulation of an algorithm to realize infinite matrix sequences with linear, constant systems represented by matrix triples. Similar algorithms were previously available (Bonivento et al., 1973; Rissanen, 1974), but several significant points are discussed here which seem to have been overlooked elsewhere.

The problem of realizing a finite matrix sequence with a linear, constant system is also considered in Chapter 2. A new partial realization algorithm which identifies the minimal number of parameters while conserving all the available degrees of freedom is given. This feature of the algorithm allows a simple parametrization of all minimal partial realizations of a given finite matrix sequence. It is shown that this algorithm has definite advantages over existing minimal partial realization algorithms (Tether, 1970; Kalman, 1971b; Ackermann, 1972).

Three non-minimal partial realization algorithms are discussed in Chapter 3. These algorithms can be effectively used to obtain all minimal partial stable realizations, all minimal partial arbitrary realizations, and all partial realizations of a specified dimension (and possibly also a desired property like decoupled structure, etc.) for a given finite matrix sequence. The approach taken in this work sheds considerable light into these problems and allows the statement of necessary conditions and/or sufficient conditions for a realization of a specified dimension and stable or arbitrary poles to exist for a given finite matrix sequence. Further, significant observations like Remark (3.35) come quite naturally with the approach followed here.

In Chapter 4 the design of intermediate order and minimal order observers to estimate vector linear functions of the state is discussed. It is shown that an observer design problem can be formulated in all cases as a partial realization problem where, generally, some columns of some of the matrices in the sequence are unspecified. Then the machinery developed in Chapters 2 and 3 is applied to the design of observers. The available results for single input and/or single output plants are extended to their limit (necessary and sufficient conditions are stated for these cases), and several new results are given for the problems of observing vector linear functions of the state of multiple output plants with minimal order observers and minimal order arbitrary observers. It is also shown that the dynamics of an observer which approximates a given feedback control law are constrained exclusively by the values of the feedback matrix in the control law and the observability indices of the plant. Several observer design techniques are examined in the light of partial realization theory and compared with the observer design proce-

dure presented here. This type of study yields considerable insight into the respective observer design techniques. The strength of the partial realization theory approach to the design of observers is demonstrated in Example (4.81).

Suggestions for Future Research

The results given in this dissertation open up several interesting possibilities for future research work. These vary from problems dealing with further polishing of various points to problems concerned with extensions of several of the results. The following problems can be immediately posed.

a) It is important to study further the relationships which exist between the number of degrees of freedom in partial realizations of finite matrix sequences and the corresponding number of arbitrary realization poles and the stability of the realizations. These problems should be examined in the light of the decidability theory developed recently by Anderson et al. (1975).

b) The question of equivalence relations and the associated invariants for the problems of obtaining minimal partial stable and minimal partial arbitrary realizations of a finite matrix sequence should be studied in more detail. The work of Morse (1972) may provide a basis for such a study.

c) It seems that a correspondence exists between the matrix-fraction description approach to observer design of Wang and Davison (1973) and the state-space approach to observer design via partial realization theory developed

here. This may lead to new applications of realization theory.

d) Several of the available observer design techniques are examined in Chapter 4 in the partial realization theory framework. It is interesting to study the correspondence (if any) with partial realization algorithms of those observer design techniques that are not considered here.

e) It remains to investigate whether the results given in this dissertation can be extended or applied in any form to the realization of stochastic systems and to the design of adaptive observers, stochastic observer-estimators, observers for nonlinear systems (particularly bilinear systems), etc.

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BIOGRAPHICAL SKETCH

Jaime Roberto Roman was born in Rio Piedras, Puerto Rico, on October 7, 1947. In May, 1965, he was graduated from San Antonio High School. In August, 1965, he entered the University of Puerto Rico, Mayaguez Campus. In May, 1970, he received the degree of Bachelor of Science in Electrical Engineering with Magna Cum Laude. He worked the summers of 1968 to 1970 for the Puerto Rico Water Resources Authority. In September, 1970, he entered the Graduate School of the University of Florida, being sponsored by a College of Engineering Fellowship until June, 1971. In December, 1971, he received the degree of Master of Science in Engineering. From January, 1972, until the present time he has done work toward the degree of Doctor of Philosophy. In March, 1975, he joined the staff of The Analytic Sciences Corporation (TASC) in Reading, Massachusetts.

While a graduate student at the University of Florida, he held various research and teaching assistantships in the Department of Electrical Engineering. His duties included research in the analysis and design of multi-rate sampled-data control systems and research in observer theory, undergraduate laboratory supervision and course grading, supervision and operation of the computer terminal facilities in the Electrical Engineering complex, and teaching an introductory course in control theory to Electrical Engineering Juniors and Seniors. At TASC he is currently involved with the application of modern control theory techniques to problems associated with inertial navigation and guidance of marine weapon systems.

Jaime Roberto Roman is married to the former Lourdes Pacheco and they have two sons, Jaime Jose and Gabriel. He is a member of Tau Beta Pi, Eta Kappa Nu, Sociedad de Ingenieros Electricistas de Puerto Rico, and the Institute of Electrical and Electronics Engineers.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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Dr. Thomas E. Bullock, Chairman
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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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